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Non-isothermal dispersed phase of particles in turbulent flow

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In this paper we consider, for modelling and simulation, a non-isothermal turbulent flow laden with non-evaporating spherical particles which exchange heat with the surrounding fluid and do not collide with each other during the course of their journey under the influence of the stochastic fluid drag force. In the modelling part of this study, a closed kinetic or probability density function (p.d.f.) equation is derived which describes the distribution of position $\boldsymbol{x}$, velocity $\boldsymbol{v}$, and temperature θ of the particles in the flow domain at time t. The p.d.f. equation represents the transport of the ensemble-average (denoted by $\langle\,\rangle$) phase-space density $\langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle$. The process of ensemble averaging generates unknown terms, namely the phase-space diffusion current $\boldsymbol{j} = \beta_v \langle \boldsymbol{u}' W\rangle$ and the phase-space heat current $h = \beta_\theta \langle t' W\rangle$, which pose closure problems in the kinetic equation. Here, $\boldsymbol{u}'$ and t' are the fluctuating parts of the velocity and temperature, respectively, of the fluid in the vicinity of the particle, and β_v and β_θ are inverse of the time constants for the particle velocity and temperature, respectively. The closure problems are first solved for the case of homogeneous turbulence with uniform mean velocity and temperature for the fluid phase by using Kraichnan's Lagrangian history direct interaction (LHDI) approximation method and then the method is generalized to the case of inhomogeneous flows. Another method, which is due to Van Kampen, is used to solve the closure problems, resulting in a closed kinetic equation identical to the equation obtained by the LHDI method. Then, the closed equation is shown to be compatible with the transformation constraint that is proposed by extending the concept of random Galilean transformation invariance to non-isothermal flows and is referred to as the 'extended random Galilean transformation' (ERGT). The macroscopic equations for the particle phase describing the time evolution of statistical properties related to particle velocity and temperature are derived by taking various moments of the closed kinetic equation. These equations are in the form of transport equations in the Eulerian framework, and are computed for the case of two-phase homogeneous shear turbulent flows with uniform temperature gradients. The predictions are compared with the direct numerical simulation (DNS) data which are generated as another part of this study. The predictions for the particle phase require statistical properties of the fluid phase which are taken from the DNS data. In DNS, the continuity, Navier–Stokes, and energy equations are solved for homogeneous turbulent flows with uniform mean velocity and temperature gradients. For the mean velocity gradient along the x_2- (cross-stream) axis, three different cases in which the mean temperature gradient is along the x_1-, x_2-, and x_3-axes, respectively, are simulated. The statistical properties related to the particle phase are obtained by computing the velocity and temperature of a large number of particles along their Lagrangian trajectories and then averaging over these trajectories. The comparisons between the model predictions and DNS results show very encouraging agreement.

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1. Introduction

Particle/droplet-laden turbulent flows have received a great deal of attention from physicists and engineers owing to their common occurrence in important natural and technological situations. Various existing analytical methods for the description of such flows can be broadly classified into two categories, namely, (i) Lagrangian and (ii) Eulerian descriptions. In both of these descriptions, the fluid (carrier) phase is described by the continuity, Navier–Stokes, and energy equations, written in the Eulerian framework. In the Lagrangian description, the particle phase is described by governing equations written for each particle in a Lagrangian framework. In the Eulerian description, the particle phase is described by 'fluid-like' equations which are derived, by some kind of averaging, from the Lagrangian equations for the particle.

In this paper, our main concern is the statistical description of the particle phase in an Eulerian framework. In this framework, the existing statistical modelling of the particle phase can be viewed as one of the two approaches, namely (i) Reynolds-averaged Navier–Stokes (RANS) modelling and (ii) kinetic equation or probability density function (p.d.f.) modelling. These modelling approaches result in continuum governing equations for the statistical properties of the particle phase.

In the RANS approach, the analysis starts from the Eulerian equations for particle instantaneous variables associated with the *cloud of particles* present in a unit volume for a single realization of the flow (Mashayek & Taulbee 2002*a*, *b*). These initial Eulerian equations are obtained by various methods of averaging (Jackson 1997; Zhang & Prosperetti 1994). The ensemble average of these equations for turbulent flows poses closure problems due to the appearance of unknown correlations similar to the Reynolds stress for fluid turbulence. Utilizing various closure schemes of fluid turbulence, different models have been proposed to close the unknown terms and have been recently reviewed by Mashayek & Pandya (2002).

In the kinetic or p.d.f. modelling approach, the analysis starts from the Lagrangian equations for individual particles and a kinetic equation is obtained which governs the probability density of particle position, velocity and other variables of interest at time t. If solved using the Monte Carlo method, this treatment would have characteristics of the Lagrangian approach. Instead of solving the p.d.f. equation, one can derive 'macroscopic' or 'fluid' equations for the dispersed phase by taking various moments of the p.d.f. equation. These equations govern the statistical properties of the particle phase and are in the Eulerian framework. The kinetic approach is also mathematically robust for the derivation of boundary conditions for the particle phase (Alipchenkov, Zaichik & Simonin 2001). Here we discuss the p.d.f. approach under the Eulerian description due to the framework of the macroscopic equations.

The p.d.f. approach has its base in the study of kinetic theory of gases (Boltzmann 1964; Chapman & Cowling 1970) and Brownian motion (Chandrasekhar 1954; Gardiner 1985) and various existing p.d.f. models have been reviewed recently by Minier & Peirano (2001). These are categorized as one-point and two-point p.d.f. models. In the one-point p.d.f. models, an equation governing the probability density of variables of interest of the particle and fluid phase along the particle path is derived. These variables are known as the 'state vector' and various one-point p.d.f. models differ in the selection of the variables of the state vector.

In the case of isothermal turbulent flow in which the particle moves under the action of the fluid drag force, the simplest choice for the variables of the state vector is the particle position and velocity. The derivation of the governing equation for the probability density function $\langle W(\boldsymbol{x}, \boldsymbol{v}, t)\rangle$ of the particle position $\boldsymbol{x}$ and velocity $\boldsymbol{v}$ at time

t requires the fluid velocity along the particle path, or 'seen' by the particle (Pozorski & Minier 1999), to be a known external variable. Here $\langle\,\rangle$ represents the ensemble average and $W(\boldsymbol{x}, \boldsymbol{v}, t)$ is the phase-space density. The equation for W can be written by using the Liouville theorem in conjunction with the Lagrangian equations governing the particle position and velocity. The ensemble average of this equation poses a closure problem due to the appearance of the unknown correlation $\langle \boldsymbol{u}' W(\boldsymbol{x}, \boldsymbol{v}, t)\rangle$ where $\boldsymbol{u}'$ is the fluctuating part of the fluid velocity along the particle path. The closed expression, after solving the closure problem, for $\langle \boldsymbol{u}' W(\boldsymbol{x}, \boldsymbol{v}, t)\rangle$ requires statistical properties related to the fluid velocity as seen by the particle (Reeks 1991, 1992; Hyland, McKee & Reeks 1999*a*; Zaichik 1999; Derevich 2000), whose predictions in general flows remain a challenge (Minier & Peirano 2001; Minier 1999).

Alternatively, the fluid velocity along the particle path is included in the state vector (Minier & Peirano 2001; Minier 1999; Pozorski & Minier 1999). In this type of p.d.f. model, writing an equation for the phase-space density $W(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}, t)$ requires a Lagrangian equation for the fluid velocity $\boldsymbol{u}$, i.e. fluid acceleration, along the particle path. And the major task is to first form this Lagrangian equation and then solve the closure problem which appears in the form of the unknown correlation $\langle \boldsymbol{a}' W(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}, t)\rangle$. Here $\boldsymbol{a}'$ is the fluctuating part of the acceleration of fluid along the particle trajectory. Pozorski & Minier (1999) modelled the fluid velocity along the particle path through a Langevin equation. This Langevin model equation giving the increment of fluid velocity along the particle path has the form of a diffusion process with a linear drift term.

In the two-point p.d.f. models, an equation is sought for the probability density function $\langle W\rangle$ of the state vector for the particle and the state vector for the fluid phase. For example, one may consider the two-point p.d.f. $\langle W(t; \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{x}_f, \boldsymbol{u}_f, \phi_f)\rangle$ which represents the probability density at time t, of the particle position $\boldsymbol{x}$, velocity $\boldsymbol{v}$, fluid velocity seen by the particle $\boldsymbol{u}$, fluid velocity $\boldsymbol{u}_f$ and its other properties ϕ_f at location $\boldsymbol{x}_f$. This approach has been discussed at greater length by Minier & Peirano (2001). Attention is focused in this present paper only on the one-point p.d.f. models with fluid properties along the particle path considered as known external variables.

In two-phase non-isothermal turbulent flow, particles move under the influence of the fluid drag force and exchange heat with the surrounding fluid. The simplest choice for the variables of the state vector is $(\boldsymbol{x}, \boldsymbol{v}, \theta)$ with fluid velocity U_i and temperature T being treated as external variables. Here θ is the phase space variable corresponding to the particle temperature T_p. When the heat transfer is driven by the temperature difference $T - T_p$, the derivation of a closed single-point p.d.f. equation for $\langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle$ requires tackling closure problems posed by the unknown term $\langle t' W\rangle$ in addition to the term $\langle \boldsymbol{u}' W\rangle$ (Zaichik 1999; Pandya & Mashayek 2002*a*), where t' is the temperature fluctuation of the fluid along the particle path.

The closure problems involved in p.d.f. approaches are similar in nature to the well-known turbulence closure problem. Therefore, it is not surprising that their solution methods, so far, have been much influenced by the turbulence closure theories. An attempt to solve the turbulence closure led Kraichnan to propose the direct interaction approximation (DIA) (Kraichnan 1958, 1959) as a pioneering renormalized perturbation theory (RPT), followed by other RPTs (Leslie 1973; McComb 1990). The failure of the energetically consistent DIA in obtaining the Kolmogorov spectrum led Kraichnan to propose the Lagrangian history direct interaction (LHDI) and the concept of random Galilean transformation invariance (Kraichnan 1965). Kraichnan's Lagrangian framework was followed by Kaneda (1981) in developing the Lagrangian renormalized approximation (LRA), for which an alternative derivation was given by

Kida & Goto (1997). These theories, DIA and LHDI, have also found applications in solving the closure problems in Vlasov plasma in the framework of the kinetic approach (Orszag & Kraichnan 1967; Orszag 1968) and have provided the foundation for Reeks' works (Reeks 1980, 1983, 1991, 1992; Reeks & McKee 1991) on the kinetic equation for the particle phase. Reeks successfully applied DIA and LHDI theories to solve the closure problem posed by $\langle \boldsymbol{u}'W\rangle$ in the kinetic equation (Reeks 1980, 1992). The LHDI closure solution is considered superior as it preserves the symmetry of the phenomena of two-phase turbulent flow under the random Galilean transformation (Reeks & McKee 1991; Reeks 1991, 1992). Recently, alternative derivations of the same closed kinetic equation were provided by Pozorski & Minier (1999) using Van Kampen's method (Van Kampen 1974*a*, *b*, 1992), and by Hyland *et al.* (1999*a*) using the Furutsu–Novikov–Donsker functional formula. This formula was first used by Derevich & Zaichik (1989, 1990) and forms the basis in their work on two-phase turbulent flows (Zaichik 1999; Zaichik *et al.* 1997*a*, *b*; Derevich 2000). Recently, Pandya & Mashayek (2002*a*) have used the functional framework of Hyland *et al.* (1999*a*) and have obtained the closed kinetic equation for a non-isothermal dispersed phase by deriving expressions for $\langle \boldsymbol{u}'W\rangle$ and $\langle t'W\rangle$.

The kinetic equation, when solved numerically with appropriate boundary and initial conditions, predicts the probability density, from which various statistical properties related to the particle phase in physical space–time $(\boldsymbol{x}, t)$ can be obtained. In another method, taking various moments of the kinetic equation provides the macroscopic equations governing the statistical properties of the particle phase in $(\boldsymbol{x}, t)$ space. These equations are in the Eulerian framework and require solving additional closure problems arising due to the process of taking the moments. For example, the macroscopic equation for the mean velocity of the particle phase $\overline{V}_i(\boldsymbol{x}, t)$ would contain the unknown term $\overline{v_i'v_j'}$ whose macroscopic equation would include the higher-order unknown term $\overline{v_i'v_j'v_k'}$, and so on. Here v_i' is a fluctuation in the particle phase velocity and the overbar represents the number-density-weighted ensemble average (for details see § 5). There are a few methods to tackle this type of closure problem (Zaichik 1999; Derevich 2000; Swailes, Sergeev & Parker 1998).

The motion of the particle phase is also affected by the compressible nature of the fluid. In compressible turbulent flows, the two new phenomena of turbulent thermal diffusion and turbulent barodiffusion of isothermal particles are proposed by Elperin, Kleeorin & Rogachevskii (1998) and were recently quantified (Pandya & Mashayek 2002*b*) in the macroscopic equations obtained by the kinetic approach. The compressibility effects of fluid on the non-isothermal particles are further studied in this paper.

It is well established that direct numerical simulation (DNS) of two-phase turbulent flows provides a rich source of data unfolding the physics of the phenomena of these flows (see Mashayek 1998, 2000 and references cited therein). The DNS studies, at present, are limited to simple flow geometries and these studies provide data, which are very difficult if not impossible to obtain experimentally, for the assessment of macroscopic equations in simple cases. As a part of this study, we generate DNS data for the case of homogeneous shear flows with uniform temperature gradients in which the particle temperature does not remain constant due to the involved heat transfer. The details of the DNS method are presented in Shotorban, Mashayek & Pandya (2002) and only the required data are presented in this paper.

In this paper, we consider non-isothermal two-phase turbulent flows in which solid spherical particles exchange heat with the carrier fluid and do not collide with each other. The specific objectives of the paper are: (i) to derive a closed kinetic equation for the non-isothermal particle phase in inhomogeneous flow by using two different methods, namely LHDI and Van Kampen's method, (ii) to propose a new transformation constraint, related to random Galilean transformation, (iii) to obtain macroscopic equations for the particle phase in an Eulerian framework, (iv) to study the effect of fluid compressibility on the particle phase through the macroscopic equations, and (v) to assess the macroscopic equations against the data obtained from DNS performed on simple homogeneous shear flow with uniform mean temperature gradients of the fluid phase. The governing equations for the particle trajectory and temperature are given in § 2. In § 3, we discuss the Liouville equations and the closure problems involved in the ensemble average of this equation. The closure problem is solved, in § 4, for homogeneous flows by using the LHDI and the method is then extended to non-homogeneous flows to propose a closed kinetic equation. Also, in § 4, an alternative derivation of the same kinetic equation is provided by using Van Kampen's method and then the closed kinetic equation is shown to be compatible with a newly proposed concept of extended random Galilean transformation (ERGT) invariance. The kinetic equation is then used in § 5 to derive the Eulerian or macroscopic equations to describe the transport of statistical properties of the particle phase. Further, algebraic relations are obtained for cross-correlations related to particle and fluid phases. The effects of the fluid compressibility on the particle phase are discussed in § 6, through the macroscopic equations. In § 7, we first give a brief account of the DNS carried out to assess the macroscopic equations and then compare the predictions of these equations to the DNS data for homogeneous shear flow with uniform mean temperature gradient. Finally, some concluding remarks are provided in § 8.

2. Lagrangian equations for the particles

For the mathematical formulation we assume point particles in a non-isothermal turbulent flow. The Lagrangian equations governing the position X_i (also denoted by vector $\boldsymbol{X}$), velocity V_i (also denoted by vector $\boldsymbol{V}$), and temperature T_p along the trajectory of each spherical particle of mass m_p and specific heat coefficient C_p can be written

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = V_i, \tag{2.1}$$

$$m_p \frac{\mathrm{d}V_i}{\mathrm{d}t} = F_i, \tag{2.2}$$

$$m_p C_p \frac{\mathrm{d}T_p}{\mathrm{d}t} = H, \tag{2.3}$$

where F_i denotes the summation of all the forces acting on the particle and H is the net rate of heat transfer to the particle. In writing (2.3), the temperature variation inside the particle is neglected and thus the particle temperature is considered uniform.

The general form for F_i which includes the fluid drag, added mass, and Basset history forces with the flow curvature effect was proposed by Maxey & Riley (1983) and still remains open for further improvement (Kim, Elghobashi & Sirignano 1998). Here, we consider only the fluid drag force acting on a particle with diameter d given by

$$F_i = \frac{C_D}{8} \pi d^2 \rho |\boldsymbol{U} - \boldsymbol{V}| (U_i - V_i), \tag{2.4}$$

where the drag coefficient, C_D, can be taken as (see Clift, Grace & Weber 1978 for a list of possible expressions for C_D)

$$C_D = \frac{24}{Re_p}(1 + 0.15 Re_p^{0.687}), \tag{2.5}$$

for particle Reynolds number $Re_p = d\rho|\boldsymbol{U} - \boldsymbol{V}|/\mu < 1000$. In these equations, $\boldsymbol{U}$ is the instantaneous velocity of the fluid, with density ρ and viscosity μ, in the vicinity of the particle.

The general expression for H has been derived by Michaelides & Feng (1994, 1996) for a rigid sphere with high thermal conductivity and at low Péclet number in unsteady velocity and temperature fields of the fluid. This expression has terms which are analogous to the drag, added mass, and history terms in the particle momentum equation. Here we consider only the term analogous to the drag, written as

$$H = Nu\pi d\kappa(T - T_p), \tag{2.6}$$

where κ is the thermal conductivity of the fluid, T is the fluid temperature in the vicinity of the particle, and the Nusselt number, Nu, for the spherical particle is given by the Ranz–Marshall correlation (Ranz & Marshall 1952)

$$Nu = 2 + 0.6 Re_p^{0.5} Pr^{0.33} \quad \forall\ Re_p < 5 \times 10^4, \tag{2.7}$$

with $Pr = C_p\mu/\kappa$ denoting the Prandtl number. Here C_p is specific heat of the fluid.

Using (2.4)–(2.7), Lagrangian equations (2.1)–(2.3) are now

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = V_i, \tag{2.8}$$

$$\frac{\mathrm{d}V_i}{\mathrm{d}t} = \beta_v(U_i - V_i), \tag{2.9}$$

$$\frac{\mathrm{d}T_p}{\mathrm{d}t} = \beta_\theta(T - T_p) + Q, \tag{2.10}$$

with

$$\beta_v = \frac{1 + 0.15 Re_p^{0.687}}{\tau_p}, \quad \tau_p = \frac{\rho_p d^2}{18\mu}, \quad \beta_\theta = \frac{2 + 0.6 Re_p^{0.5} Pr^{0.33}}{3\tau_p Pr\sigma}, \tag{2.11}$$

where τ_p is the particle momentum relaxation time or particle time constant, $\sigma = C_p/C_f$, C_f is the specific heat of the fluid and we have also added a source term Q for the particle temperature in (2.10). These Lagrangian equations (2.8)–(2.10) form the basis of our derivation of the single-point p.d.f. equation.

3. The Liouville equation and closure problems

The main steps in obtaining a single-point p.d.f. equation are (i) forming an equation for the phase-space density W and (ii) obtaining the closed form for the ensemble average of the phase-space density equation by solving the closure problem involved. In the present case, we seek to form an equation for $W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)$ using the Liouville theorem in conjunction with the Lagrangian equations (2.8)–(2.10). The 'fine grained' phase-space density $W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)$ is defined using the Dirac delta function as (Pope 1985; Hyland *et al.* 1999*a*)

$$W(\boldsymbol{x}, \boldsymbol{v}, \theta, t) = \delta(\boldsymbol{X} - \boldsymbol{x})\delta(\boldsymbol{V} - \boldsymbol{v})\delta(T_p - \theta), \tag{3.1}$$

and its ensemble average $\langle\ \rangle$ over all the realizations represents the probability density function $\langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle$. Here, $\boldsymbol{x}, \boldsymbol{v}$ and θ are phase-space variables corresponding to $\boldsymbol{X}, \boldsymbol{V}$ and T_p, respectively. The governing equation for W for collisionless particles can be obtained from the Liouville theorem (or following the procedure given in Hyland *et al.* 1999*a*) and using the Lagrangian equations (2.8)–(2.10), and is

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_i}[v_i W] + \frac{\partial}{\partial v_i}[\beta_v (U_i - v_i) W] + \frac{\partial}{\partial \theta}[\beta_\theta (T - \theta) W + Q W] = 0. \tag{3.2}$$

In (3.2), β_v and β_θ are stochastic functions in the case of turbulent flows due to the fluctuations in U_i, and W is also stochastic in nature. The ensemble average of (3.2),

$$\frac{\partial \langle W\rangle}{\partial t} + \frac{\partial}{\partial x_i}[v_i \langle W\rangle] + \frac{\partial}{\partial v_i}[\langle\beta_v\rangle(\langle U_i\rangle - v_i)\langle W\rangle] + \frac{\partial}{\partial \theta}[\langle\beta_\theta\rangle(\langle T\rangle - \theta)\langle W\rangle + Q\langle W\rangle]$$
$$= -\frac{\partial}{\partial v_i}[\langle\beta_v\rangle\langle u_i' W\rangle] - \frac{\partial}{\partial \theta}[\langle\beta_\theta\rangle\langle t' W\rangle], \tag{3.3}$$

poses closure problems due to the unknown correlations $\langle u_i' W\rangle$ and $\langle t' W\rangle$ present on the right-hand side of (3.3). Here, u_i' and t' are the fluctuating parts of $U_i = \langle U_i\rangle + u_i'$ and $T = \langle T\rangle + t'$ over the average value $\langle U_i\rangle$ and $\langle T\rangle$, respectively. Also, in writing (3.3), terms containing fluctuating parts β_v' and β_θ' are neglected, and from now onwards we assume $\langle\beta_v\rangle$ and $\langle\beta_\theta\rangle$ to be weak functions of $\langle Re_p\rangle$ and consider them as constants during the analysis; thus they are simply written as β_v and β_θ. In effect we are considering

$$\beta_v = \frac{1 + 0.15\langle Re_p^{0.687}\rangle}{\tau_p}, \quad \beta_\theta = \frac{2 + 0.6\langle Re_p^{0.5}\rangle Pr^{0.33}}{3\tau_p Pr\sigma} \tag{3.4}$$

in the Lagrangian equations (2.9) and (2.10).

In case of isothermal two-phase flow, (3.2) and (3.3) reduce to

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_i}[v_i W] + \frac{\partial}{\partial v_i}[\beta_v (U_i - v_i) W] = 0 \tag{3.5}$$

for $W(\boldsymbol{x}, \boldsymbol{v}, t)$, and

$$\frac{\partial \langle W\rangle}{\partial t} + \frac{\partial}{\partial x_i}[v_i \langle W\rangle] + \frac{\partial}{\partial v_i}[\beta_v(\langle U_i\rangle - v_i)\langle W\rangle] = -\frac{\partial}{\partial v_i}[\beta_v \langle u_i' W\rangle] \tag{3.6}$$

for $\langle W(x_i, v_i, t)\rangle$, respectively. The closure problem in this case is due to the unknown correlation $\langle u_i' W\rangle$ present on the right-hand side of (3.6). The unknown term $J_i = \beta_v \langle u_i' W\rangle$ is known as the phase space diffusion current (Reeks 1991). The unknown term $h = \beta_\theta \langle t' W\rangle$ in (3.3) is referred to as the phase-space heat current (Pandya & Mashayek 2002*a*). The closed expressions for J_i and h are obtained by Zaichik (1999) and Pandya & Mashayek (2002*a*) in the context of functional formalism using the Furutsu–Novikov–Donsker formula. These expressions are exact in cases where u_i' and t' are Gaussian random functions. In the next section, we use two different methods, namely LHDI and Van Kampen's method, to solve the closure problems as these methods are *not* limited to the case where u_i' and t' are Gaussian functions.

4. Solutions to the closure problems

In the present context, the motivation for using LHDI and Van Kampen's method is their success in obtaining the closed kinetic equations for isothermal particle phase (Reeks 1992; Pozorski & Minier 1999). These methods belong to the class of renormalized perturbation techniques, and, in a renormalized perturbation series, the lowest order closure solution obtained for $\langle \boldsymbol{u}' W(\boldsymbol{x}, \boldsymbol{v}, t)\rangle$ by these methods (Reeks 1992; Pozorski & Minier 1999) is identical to the solution obtained in the functional formalism (Hyland *et al.* 1999*a*). These two methods are not restricted by the requirement of Gaussian distributions for fluid flow variables along the particle path which is the basis of the application of the Furutsu–Novikov–Donsker formula in functional formalism (Zaichik 1999; Hyland *et al.* 1999*a*; Derevich 2000; Pandya & Mashayek 2002*a*). In this section, the closure problems posed by unknown terms J_i and h are tackled and solved in the framework of LHDI and Van Kampen's method. Then, we propose a new transformation constraint of extended random Galilean transformation (ERGT) for non-isothermal flows and show that the obtained closure solutions satisfy *exactly* the ERGT invariance properties.

4.1. *Application of LHDI*

4.1.1. *Dispersed phase in turbulent flows with uniform mean*

Here, we use LHDI to obtain the p.d.f. equation for the particle phase in turbulent flows with constant values for fluid mean velocity $\langle U_i\rangle$ and mean temperature $\langle T\rangle$. For the sake of simplicity, we assume the temperature source term $Q = 0$. For this case, (3.2) becomes

$$\frac{\partial}{\partial t}W(\boldsymbol{x},\boldsymbol{v},\theta,t)+\frac{\partial}{\partial x_i}[v_iW]+\frac{\partial}{\partial v_i}[\beta_v(\langle U_i\rangle+u_i'-v_i)W]+\frac{\partial}{\partial \theta}[\beta_\theta(\langle T\rangle+t'-\theta)W]=0. \quad (4.1)$$

Equation (4.1) is a linear equation for W and its solution for known initial values for $W(\boldsymbol{x}_1, \boldsymbol{v}_1, \theta_1, t_0)$ at time $t = t_0$ is

$$W(\boldsymbol{x},\boldsymbol{v},\theta,t)=\int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\theta_1 \hat{G}(\boldsymbol{x},\boldsymbol{v},\theta,t;\boldsymbol{x}_1,\boldsymbol{v}_1,\theta_1,t_0)W(\boldsymbol{x}_1,\boldsymbol{v}_1,\theta_1,t_0), \quad (4.2)$$

where $\hat{G}$ is Green's function. An ensemble average of (4.2), when $\hat{G}$ is not correlated with initial values of W, is

$$\langle W(\boldsymbol{x},\boldsymbol{v},\theta,t)\rangle=\int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\theta_1 G(\boldsymbol{x},\boldsymbol{v},\theta,t;\boldsymbol{x}_1,\boldsymbol{v}_1,\theta_1,t_0)\langle W(\boldsymbol{x}_1,\boldsymbol{v}_1,\theta_1,t_0)\rangle, \quad (4.3)$$

where $G = \langle\hat{G}\rangle$ is the average Green's function. Equations (4.2)–(4.3) suggest that the solution for W can be completely given by Green's function when initial conditions for W are known. Green's function satisfies

$$\frac{\partial}{\partial t}\hat{G}(\boldsymbol{x},\boldsymbol{v},\theta,t;\boldsymbol{x}_1,\boldsymbol{v}_1,\theta_1,t_1)+\frac{\partial}{\partial x_i}[v_i\hat{G}]+\frac{\partial}{\partial v_i}[\beta_v(\langle U_i\rangle+u_i'-v_i)\hat{G}] +\frac{\partial}{\partial\theta}[\beta_\theta(\langle T\rangle+t'-\theta)\hat{G}]=0 \quad (4.4)$$

$\forall\ t > t_1$, and $\hat{G}(\boldsymbol{x},\boldsymbol{v},\theta,t;\boldsymbol{x}_1,\boldsymbol{v}_1,\theta_1,t_1) = \delta(\boldsymbol{x}-\boldsymbol{x}_1)\delta(\boldsymbol{v}-\boldsymbol{v}_1)\delta(\theta-\theta_1)$ when $t = t_1$. Now, the ensemble average of (4.4),

$$\frac{\partial}{\partial t}G(\boldsymbol{x},\boldsymbol{v},\theta,t;\boldsymbol{x}_1,\boldsymbol{v}_1,\theta_1,t_1)+\frac{\partial}{\partial x_i}[v_iG]+\frac{\partial}{\partial v_i}[\beta_v(\langle U_i\rangle-v_i)G]+\frac{\partial}{\partial\theta}[\beta_\theta(\langle T\rangle-\theta)G] =-\frac{\partial}{\partial v_i}\beta_v\langle u_i'\hat{G}\rangle-\frac{\partial}{\partial\theta}\beta_\theta\langle t'\hat{G}\rangle, \quad (4.5)$$

poses closure problems due to unknown correlations $\langle u_i'\hat{G}\rangle$ and $\langle t'\hat{G}\rangle$, for which closed expressions are obtained here via LHDI.

Before applying LHDI, we transform (4.4) to a new phase space (y_i, w_i, ϕ, t), using the following transformations:

$$y_i = x_i + \frac{v_i}{\beta_v}(1 - \mathrm{e}^{\beta_v t}) - \frac{\langle U_i\rangle}{\beta_v}(1 - \mathrm{e}^{\beta_v t}) - \langle U_i\rangle t, \tag{4.6}$$

$$w_i = v_i\,\mathrm{e}^{\beta_v t} + \langle U_i\rangle(1 - \mathrm{e}^{\beta_v t}), \tag{4.7}$$

$$\phi = \theta\,\mathrm{e}^{\beta_\theta t} + \langle T\rangle(1 - \mathrm{e}^{\beta_\theta t}); \tag{4.8}$$

it is written as

$$\frac{\partial}{\partial t}\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1) = l_i(\boldsymbol{y}, \boldsymbol{w}, \phi, t)[\beta_v u_i'(\boldsymbol{x}, t)\hat{G}] + M(\boldsymbol{y}, \boldsymbol{w}, \phi, t)[\beta_\theta t'(\boldsymbol{x}, t)\hat{G}], \tag{4.9}$$

where operators l_i and M are defined as

$$l_i(\boldsymbol{y}, \boldsymbol{w}, \phi, t) = -\frac{(1 - \mathrm{e}^{\beta_v t})}{\beta_v}\frac{\partial}{\partial y_i} - \mathrm{e}^{\beta_v t}\frac{\partial}{\partial w_i}, \tag{4.10}$$

$$M(\boldsymbol{y}, \boldsymbol{w}, \phi, t) = -\mathrm{e}^{\beta_\theta t}\frac{\partial}{\partial \phi}, \tag{4.11}$$

and for the three-dimensional case we have

$$\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1) = \exp\left(-(3\beta_v + \beta_\theta)(t - t_1)\right)\hat{G}(\boldsymbol{x}, \boldsymbol{v}, \theta, t; \boldsymbol{x}_1, \boldsymbol{v}_1, \theta_1, t_1). \tag{4.12}$$

It follows from (4.9) that $\hat{G}$ and consequently the phase-space density W and the p.d.f. $\langle W\rangle$ remain constant in the new phase space in the absence of turbulent fluctuations in fluid velocity u_i' and temperature t'. Also, the Lagrangian equations for the particle

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = u_i'(1 - \mathrm{e}^{\beta_v t}), \tag{4.13}$$

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} = \beta_v\,\mathrm{e}^{\beta_v t}u_i', \tag{4.14}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \beta_\theta\,\mathrm{e}^{\beta_\theta t}t', \tag{4.15}$$

suggest that the particle remains at rest under the transformations (4.6)–(4.8) when $u_i' = t' = 0$.

Now we use the method of LHDI to obtain a closed equation, from (4.9), for the ensemble average of $\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1)$ and then the resulting equation is transformed back to the original phase space (x_i, v_i, θ, t) to obtain the required closed equation for $G(\boldsymbol{x}, \boldsymbol{v}, \theta, t; \boldsymbol{x}_1, \boldsymbol{v}_1, \theta_1, t_1)$.

Following LHDI, we introduce the generalized Green's function $\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1)$. This represents the response of the phase-space density $W(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s)$ to the perturbation which is applied at time s_1 to a particle with a trajectory passing through $(\boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1)$ at time t_1. The argument $(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s)$ of $\hat{G}$ means that the response is measured at time s along the particle trajectory which passes through $(\boldsymbol{y}, \boldsymbol{w}, \phi)$ at time t. The only restriction required is on measuring times (s, s_1) such that $s > s_1$. We also introduce the generalized functions $u_i'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s)$ and $t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s)$ which represent the fluctuating part of the fluid velocity and temperature measured at time s along the particle trajectory which passes through $(\boldsymbol{x}, \boldsymbol{v}, \theta)$ at time t. In LHDI terminology,

t is the 'labelling time' and s is the 'measuring time'. For $t \neq s$, $u_i'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s)$ and $t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s)$ have Lagrangian characteristics, and for $t = s$

$$u_i'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s) = u_i'(\boldsymbol{x}, t), \quad t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s) = t'(\boldsymbol{x}, t). \tag{4.16}$$

The form of the equation for the generalized Green's function remains the same as that of equation (4.9) (Reeks 1992) and for $s > s_1$ is written as

$$\begin{aligned} \frac{\partial}{\partial t}\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1) \\ = \lambda l_i(\boldsymbol{y}, \boldsymbol{w}, \phi, t)[\beta_v u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t)\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1)] \\ +\lambda M(\boldsymbol{y}, \boldsymbol{w}, \phi, t)[\beta_\theta t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t)\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1)], \end{aligned} \tag{4.17}$$

where we have replaced $\hat{G}$ in (4.9) by the generalized Green's function and have used

$$u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t) = u_i'(\boldsymbol{x}, t), \quad t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t) = t'(\boldsymbol{x}, t). \tag{4.18}$$

In (4.17), we have introduced the usual perturbation expansion parameter λ for future convenience and which is set equal to one at the end of the analysis.

For $t \neq s$, the governing equations for $u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s)$ and $t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s)$ can be obtained by using (4.13)–(4.15) and the fact that these velocity and temperature fields measured at time s are independent of the particular point along the chosen phase-space trajectory passing through $(\boldsymbol{y}, \boldsymbol{w}, \phi, t)$. Mathematically, this is

$$u_i'(\boldsymbol{y} + \delta\boldsymbol{y}, \boldsymbol{w} + \delta\boldsymbol{w}, \phi + \delta\phi, t + \delta t|s) = u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s), \tag{4.19}$$

$$t'(\boldsymbol{y} + \delta\boldsymbol{y}, \boldsymbol{w} + \delta\boldsymbol{w}, \phi + \delta\phi, t + \delta t|s) = t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s), \tag{4.20}$$

where $\delta\boldsymbol{y}, \delta\boldsymbol{w}, \delta\phi$, and δt are small increments along the chosen trajectory. The final equations when $t \neq s$ are

$$\begin{aligned} \frac{\partial}{\partial t}u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s) = \lambda\beta_v u_j'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t) l_j(\boldsymbol{y}, \boldsymbol{w}, \phi, t) u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s) \\ +\lambda\beta_\theta t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t) M(\boldsymbol{y}, \boldsymbol{w}, \phi, t) u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s), \end{aligned} \tag{4.21}$$

$$\begin{aligned} \frac{\partial}{\partial t}t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s) = \lambda\beta_v u_j'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t) l_j(\boldsymbol{y}, \boldsymbol{w}, \phi, t) t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s) \\ +\lambda\beta_\theta t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t) M(\boldsymbol{y}, \boldsymbol{w}, \phi, t) t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s). \end{aligned} \tag{4.22}$$

In (4.21)–(4.22), we have again introduced the perturbation parameter λ which is set equal to one at the end of the analysis.

The ensemble average of (4.17),

$$\begin{aligned} \frac{\partial}{\partial t}G(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1) \\ = \lambda l_i(\boldsymbol{y}, \boldsymbol{w}, \phi, t)[\beta_v \langle u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t)\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1)\rangle] \\ +\lambda M(\boldsymbol{y}, \boldsymbol{w}, \phi, t)[\beta_\theta \langle t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t)\hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1)\rangle], \end{aligned} \tag{4.23}$$

poses closure problems due to the unknown correlations $\langle u_i'\hat{G}\rangle$ and $\langle t'\hat{G}\rangle$, for which we now obtain closed expressions. Also, equations (4.17) and (4.23) suggest that $\hat{G}$ and G are independent of the measuring time. We expand the generalized Green's function, $u_i'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t)$, and $t'(\boldsymbol{y}, \boldsymbol{w}, \phi, t|t)$ in powers of λ:

$$\begin{aligned} \hat{G}(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1) = G^{(0)}(\boldsymbol{y}, \boldsymbol{w}, \phi, t|s; \boldsymbol{y}_1, \boldsymbol{w}_1, \phi_1, t_1|s_1) \\ +\lambda\hat{G}^{(1)} + \lambda^2\hat{G}^{(2)} + \cdots, \end{aligned} \tag{4.24}$$

$$u_i'(\boldsymbol{y},\boldsymbol{w},\phi,t|s) = u_i'^{(0)}(\boldsymbol{y},\boldsymbol{w},\phi,t|s) + \lambda u_i'^{(1)} + \lambda^2 u_i'^{(2)} + \cdots, \tag{4.25}$$

$$t'(\boldsymbol{y},\boldsymbol{w},\phi,t|s) = t'^{(0)}(\boldsymbol{y},\boldsymbol{w},\phi,t|s) + \lambda t'^{(1)} + \lambda^2 t'^{(2)} + \cdots. \tag{4.26}$$

Substituting these expansions in (4.17), (4.21) and (4.22), and equating the terms with like powers in λ, we find that at zeroth order in λ

$$\frac{\partial}{\partial t} G^{(0)}(\boldsymbol{y},\boldsymbol{w},\phi,t|s;\boldsymbol{y}_1,\boldsymbol{w}_1,\phi_1,t_1|s_1) = 0, \tag{4.27}$$

$$\frac{\partial}{\partial t} u_i'^{(0)}(\boldsymbol{y},\boldsymbol{w},\phi,t|s) = 0, \tag{4.28}$$

$$\frac{\partial}{\partial t} t'^{(0)}(\boldsymbol{y},\boldsymbol{w},\phi,t|s) = 0. \tag{4.29}$$

At first order in λ, the equation for the generalized Green's function is obtained as

$$\begin{aligned}\frac{\partial}{\partial t}\hat{G}^{(1)}(\mathbf{1},t|s;\mathbf{2},t_1|s_1) = {} & l_i(\mathbf{1},t)[\beta_v u_i'^{(0)}(\mathbf{1},t|t)G^{(0)}(\mathbf{1},t|s;\mathbf{2},t_1|s_1)] \\ & + M(\mathbf{1},t)[\beta_\theta t'^{(0)}(\mathbf{1},t|t)G^{(0)}(\mathbf{1},t|s;\mathbf{2},t_1|s_1)],\end{aligned} \tag{4.30}$$

where we have introduced the **1** and **2** notation for arguments $(\boldsymbol{y},\boldsymbol{w},\phi)$ and $(\boldsymbol{y}_1,\boldsymbol{w}_1,\phi_1)$, respectively. Similarly, equations for the generalized Green's function and the generalized u_i' and t' can be written for higher orders in λ.

Equations (4.27)–(4.29) suggest that $G^{(0)}$, $u_i'^{(0)}$ and $t'^{(0)}$ are independent of labelling times, and the solution for $G^{(0)}$ is

$$G^{(0)}(\boldsymbol{y},\boldsymbol{w},\phi,t|s;\boldsymbol{y}_1,\boldsymbol{w}_1,\phi_1,t_1|s_1) = \delta(\boldsymbol{y}-\boldsymbol{y}_1)\delta(\boldsymbol{w}-\boldsymbol{w}_1)\delta(\phi-\phi_1). \tag{4.31}$$

Integrating (4.30) in the standard LHDI manner along the path $t_1|s_1 \to s_1|s_1 \to s|s \to t|s$, we have

$$\begin{aligned}\hat{G}^{(1)}(\mathbf{1},t|s;\mathbf{2},t_1|s_1) = {} & \int_{t_1}^{s_1}\!\!\int \mathrm{d}t_2\,\mathrm{d}\mathbf{3}G^{(0)}(\mathbf{1},t|s;\mathbf{3},t_2|s_1)[l_i(\mathbf{3},t_2)\beta_v u_i'^{(0)}(\mathbf{3},t_2|t_2) \\ & \times G^{(0)}(\mathbf{3},t_2|s_1;\mathbf{2},t_1|s_1) + M(\mathbf{3},t_2)\beta_\theta t'^{(0)}(\mathbf{3},t_2|t_2)G^{(0)}(\mathbf{3},t_2|s_1;\mathbf{2},t_1|s_1)] \\ & + \int_{s_1}^{s}\!\!\int \mathrm{d}t_2\,\mathrm{d}\mathbf{3}G^{(0)}(\mathbf{1},t|s;\mathbf{3},t_2|t_2)[l_i(\mathbf{3},t_2)\beta_v u_i'^{(0)}(\mathbf{3},t_2|t_2) \\ & \times G^{(0)}(\mathbf{3},t_2|t_2;\mathbf{2},t_1|s_1) + M(\mathbf{3},t_2)\beta_\theta t'^{(0)}(\mathbf{3},t_2|t_2)G^{(0)}(\mathbf{3},t_2|t_2;\mathbf{2},t_1|s_1)] \\ & + \int_{s}^{t}\!\!\int \mathrm{d}t_2\,\mathrm{d}\mathbf{3}G^{(0)}(\mathbf{1},t|s;\mathbf{3},t_2|s)[l_i(\mathbf{3},t_2)\beta_v u_i'^{(0)}(\mathbf{3},t_2|t_2) \\ & \times G^{(0)}(\mathbf{3},t_2|s;\mathbf{2},t_1|s_1) + M(\mathbf{3},t_2)\beta_\theta t'^{(0)}(\mathbf{3},t_2|t_2)G^{(0)}(\mathbf{3},t_2|s;\mathbf{2},t_1|s_1)].\end{aligned} \tag{4.32}$$

Similarly, solutions for higher-order $\hat{G}^{(n)}$ can be written in terms of $G^{(0)}$, $u_i'^{(0)}$ and $t'^{(0)}$, and symbolically presented as $F_n(G^{(0)},u_i'^{(0)},t'^{(0)})$ where $n = 1,2,3,\ldots$. Therefore, perturbation series (4.24) can be written

$$\hat{G} = G^{(0)} + \lambda F_1(G^{(0)},u_i'^{(0)},t'^{(0)}) + \lambda^2 F_2(G^{(0)},u_i'^{(0)},t'^{(0)}) + \cdots. \tag{4.33}$$

The ensemble average of (4.33) gives $G = \langle\hat{G}\rangle$ as a perturbation series having terms containing $G^{(0)}$ and various moments of $u_i'^{(0)}$ and $t'^{(0)}$. This perturbation series can be inverted using the method proposed by Kraichnan (1977) and $G^{(0)}$ can be written as a series in which each term contains G and various moments of $u_i'^{(0)}$ and $t'^{(0)}$. The lowest-order term in the series equation for $G^{(0)}$ is equal to G, i.e.

$$G^{(0)} = G + \lambda J_1 + \lambda^2 J_2 + \cdots, \tag{4.34}$$

where J_n, $n = 1, 2, \ldots$, denote higher-order terms containing G and moments of $u_i'^{(0)}$ and $t'^{(0)}$.

Now, the use of the perturbation expansion given by (4.24) in $\langle u_i' \hat{G} \rangle$ and $\langle t' \hat{G} \rangle$ allows us to write the perturbation series solutions as

$$\begin{aligned}\langle u_i'(\mathbf{1}, t|t)\hat{G}(\mathbf{1}, t|s; \mathbf{2}, t_1|s_1)\rangle &= \langle u_i'(\mathbf{1}, t|t)G^{(0)}(\mathbf{1}, t|s; \mathbf{2}, t_1|s_1)\rangle \\ &\quad + \lambda\langle u_i'(\mathbf{1}, t|t)\hat{G}^{(1)}(\mathbf{1}, t|s; \mathbf{2}, t_1|s_1)\rangle + \cdots,\end{aligned} \tag{4.35}$$

$$\begin{aligned}\langle t'(\mathbf{1}, t|t)\hat{G}(\mathbf{1}, t|s; \mathbf{2}, t_1|s_1)\rangle &= \langle t'(\mathbf{1}, t|t)G^{(0)}(\mathbf{1}, t|s; \mathbf{2}, t_1|s_1)\rangle \\ &\quad + \lambda\langle t'(\mathbf{1}, t|t)\hat{G}^{(1)}(\mathbf{1}, t|s; \mathbf{2}, t_1|s_1)\rangle + \cdots,\end{aligned} \tag{4.36}$$

where the first term on the right-hand side of (4.35) and (4.36) is equal to zero as $G^{(0)}$ is not a stochastic function which is followed by (4.27). In (4.35) and (4.36), substituting for $\hat{G}^{(1)}$ from (4.32), averaging, changing every labelling time from t_2 to t in every $G^{(0)}$, $u_i^{(0)}$ and $t'^{(0)}$ (Reeks 1992) and then replacing $G^{(0)}$ by the series given by (4.34), we obtain at the lowest order

$$\begin{aligned}&\langle u_i'(\mathbf{1}, t|t)\hat{G}(\mathbf{1}, t|s; \mathbf{2}, t_1|s_1)\rangle \\ &= \int_{s_1}^{s}\!\!\int \mathrm{d}t_2\, \mathrm{d}\mathbf{3}\, G(\mathbf{1}, t|s; \mathbf{3}, t|t_2)[\langle u_i'(\mathbf{1}, t|t) l_j(\mathbf{3}, t_2)\beta_v u_j'^{(0)}(\mathbf{3}, t|t_2)\rangle G(\mathbf{3}, t|t_2; \mathbf{2}, t_1|s_1) \\ &\quad + \langle u_i'(\mathbf{1}, t|t) M(\mathbf{3}, t_2)\beta_\theta t'^{(0)}(\mathbf{3}, t|t_2)\rangle G(\mathbf{3}, t|t_2; \mathbf{2}, t_1|s_1)] \\ &\quad + \text{two other similar terms},\end{aligned} \tag{4.37}$$

$$\begin{aligned}&\langle t'(\mathbf{1}, t|t)\hat{G}(\mathbf{1}, t|s; \mathbf{2}, t_1|s_1)\rangle \\ &= \int_{s_1}^{s}\!\!\int \mathrm{d}t_2\, \mathrm{d}\mathbf{3}\, G(\mathbf{1}, t|s; \mathbf{3}, t|t_2)[\langle t'(\mathbf{1}, t|t) l_j(\mathbf{3}, t_2)\beta_v u_j'^{(0)}(\mathbf{3}, t|t_2)\rangle G(\mathbf{3}, t|t_2; \mathbf{2}, t_1|s_1) \\ &\quad + \langle t'(\mathbf{1}, t|t) M(\mathbf{3}, t_2)\beta_\theta t'^{(0)}(\mathbf{3}, t|t_2)\rangle G(\mathbf{3}, t|t_2; \mathbf{2}, t_1|s_1)] \\ &\quad + \text{two other similar terms}.\end{aligned} \tag{4.38}$$

Substituting (4.37)–(4.38) in (4.23) and for $\lambda = 1$, $s = t$ and $s_1 = t_1$, we obtain an equation for $G(\mathbf{1}, t; \mathbf{2}, t_1) = G(\mathbf{1}, t|t; \mathbf{2}, t_1|t_1)$ as

$$\begin{aligned}\frac{\partial}{\partial t}G(\mathbf{1}, t|t; \mathbf{2}, t_1|t_1) &= l_i(\mathbf{1}, t)\int_{t_1}^{t} \mathrm{d}t_2[\langle \beta_v u_i'(\mathbf{1}, t|t) l_j(\mathbf{1}, t_2)\beta_v u_j'^{(0)}(\mathbf{1}, t|t_2)\rangle G(\mathbf{1}, t|t; \mathbf{2}, t_1|t_1) \\ &\quad + \langle \beta_v u_i'(\mathbf{1}, t|t) M(\mathbf{1}, t_2)\beta_\theta t'^{(0)}(\mathbf{1}, t|t_2)\rangle G(\mathbf{1}, t|t; \mathbf{2}, t_1|t_1)] \\ &\quad + M(\mathbf{1}, t)\int_{t_1}^{t} \mathrm{d}t_2[\langle \beta_\theta t'(\mathbf{1}, t|t) l_j(\mathbf{1}, t_2)\beta_v u_j'^{(0)}(\mathbf{1}, t|t_2)\rangle G(\mathbf{1}, t|t; \mathbf{2}, t_1|t_1) \\ &\quad + \langle \beta_\theta t'(\mathbf{1}, t|t) M(\mathbf{1}, t_2)\beta_\theta t'^{(0)}(\mathbf{1}, t|t_2)\rangle G(\mathbf{1}, t|t; \mathbf{2}, t_1|t_1)].\end{aligned} \tag{4.39}$$

In writing (4.39) we have used

$$G(\mathbf{1}, t|t; \mathbf{3}, t|t_2) = G(\mathbf{1}, t|t; \mathbf{3}, t|t) = \delta(\mathbf{1} - \mathbf{3}), \tag{4.40}$$

$$G(\mathbf{1}, t|t_2; \mathbf{2}, t_1|t_1) = G(\mathbf{1}, t|t; \mathbf{2}, t_1|t_1), \tag{4.41}$$

as G is independent of the measuring time. The right-hand side of (4.39) is still in terms of the primitive variables $u_i'(0)$ and $t'(0)$. For renormalization we replace $u_i'(0)$ and $t'(0)$ by u_i' and t' in (4.39), transform it back to the original phase space $(\boldsymbol{x}, \boldsymbol{v}, \theta, t)$, multiply the resulting equation by the initial value $\langle W(\boldsymbol{x}_1, \boldsymbol{v}_1, \theta_1, 0)\rangle$ at time $t_1 = 0$, integrate over the phase-space variables $(\boldsymbol{x}_1, \boldsymbol{v}_1, \theta_1)$, and use (4.3) to obtain a closed transport equation for the p.d.f. $\langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle$:

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_i} v_i + \frac{\partial}{\partial v_i}\beta_v(\langle U_i\rangle - v_i) + \frac{\partial}{\partial \theta}\beta_\theta(\langle T\rangle - \theta)\right)\langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle \\
&\quad = \frac{\partial}{\partial v_i}\left[\frac{\partial}{\partial x_j}\left\{\int_0^t \mathrm{d}t_2 \frac{(1 - \mathrm{e}^{-\beta_v(t-t_2)})}{\beta_v}\langle \beta_v^2 u_i'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle\right\}\right. \\
&\qquad + \frac{\partial}{\partial v_j}\left\{\int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_v(t-t_2)}\langle \beta_v^2 u_i'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle\right\} \\
&\qquad + \frac{\partial}{\partial \theta}\left\{\int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_\theta(t-t_2)}\langle \beta_v\beta_\theta u_i'(\boldsymbol{x}, t) t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle\right\} \\
&\qquad \left. - \int_0^t \mathrm{d}t_2 \frac{(1 - \mathrm{e}^{-\beta_v(t-t_2)})}{\beta_v}\left\langle \frac{\partial \beta_v u_i'(\boldsymbol{x}, t)}{\partial x_j}\beta_v u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\right\rangle\right]\langle W\rangle \\
&\qquad + \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial x_j}\left\{\int_0^t \mathrm{d}t_2 \frac{(1 - \mathrm{e}^{-\beta_v(t-t_2)})}{\beta_v}\langle \beta_v\beta_\theta t'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle\right\}\right. \\
&\qquad + \frac{\partial}{\partial v_j}\left\{\int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_v(t-t_2)}\langle \beta_v\beta_\theta t'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle\right\} \\
&\qquad + \frac{\partial}{\partial \theta}\left\{\int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_\theta(t-t_2)}\langle \beta_\theta\beta_\theta t'(\boldsymbol{x}, t) t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle\right\} \\
&\qquad \left. - \int_0^t \mathrm{d}t_2 \frac{(1 - \mathrm{e}^{-\beta_v(t-t_2)})}{\beta_v}\left\langle \frac{\partial \beta_\theta t'(\boldsymbol{x}, t)}{\partial x_j}\beta_v u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\right\rangle\right]\langle W\rangle \\
&\quad \equiv -\frac{\partial}{\partial v_i}\langle \beta_v u_i' W\rangle - \frac{\partial}{\partial \theta}\langle \beta_\theta t' W\rangle.
\end{aligned}
\tag{4.42}
$$

The right-hand side of (4.42) gives the expressions for $\langle \beta_v u_i' W\rangle$ and $\langle \beta_\theta t' W\rangle$, which can be written in the form

$$\langle \beta_v u_i' W\rangle = -\left[\frac{\partial}{\partial x_k}\lambda_{ki} + \frac{\partial}{\partial v_k}\mu_{ki} + \frac{\partial}{\partial \theta}\omega_i - \gamma_i\right]\langle W\rangle, \tag{4.43}$$

$$\langle \beta_\theta t' W\rangle = -\left[\frac{\partial}{\partial x_k}\Lambda_k + \frac{\partial}{\partial v_k}\Pi_k + \frac{\partial}{\partial \theta}\Omega - \Gamma\right]\langle W\rangle, \tag{4.44}$$

where

$$\lambda_{ki} = \langle \Delta x_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)\beta_v u_i'(\boldsymbol{x}, t)\rangle, \quad \mu_{ki} = \langle \Delta v_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)\beta_v u_i'(\boldsymbol{x}, t)\rangle, \tag{4.45}$$

$$\gamma_i = \left\langle \Delta x_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)\frac{\partial \beta_v u_i'(\boldsymbol{x}, t)}{\partial x_k}\right\rangle, \quad \omega_i = \langle \Delta\theta(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)\beta_v u_i'(\boldsymbol{x}, t)\rangle, \tag{4.46}$$

$$\Lambda_k = \langle \Delta x_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)\beta_\theta t'(\boldsymbol{x}, t)\rangle, \quad \Pi_k = \langle \Delta v_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)\beta_\theta t'(\boldsymbol{x}, t)\rangle, \tag{4.47}$$

$$\Gamma = \left\langle \Delta x_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)\frac{\partial \beta_\theta t'(\boldsymbol{x}, t)}{\partial x_k}\right\rangle, \quad \Omega = \langle \Delta\theta(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)\beta_\theta t'(\boldsymbol{x}, t)\rangle. \tag{4.48}$$

Hereafter, these tensors are referred to as ‘various tensors’. In these equations, $\Delta x_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)$ and $\Delta v_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)$ are changes in the position and velocity of the particle due to the fluid fluctuating force per unit mass, $\beta_v u_i'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)$, along the trajectory of the particle starting at time zero and passing through $(\boldsymbol{x}, \boldsymbol{v}, \theta)$ at time t. Similarly, $\Delta\theta(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)$ represents the change in the temperature of the particle due to $\beta_\theta t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)$ along the trajectory of the particle starting at time zero and passing through $(\boldsymbol{x}, \boldsymbol{v}, \theta)$ at time t. These changes in position velocity, and temperature are given by

$$\Delta x_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) = \int_0^t \mathrm{d}t_2 \frac{(1 - \mathrm{e}^{-\beta_v(t-t_2)})}{\beta_v} \beta_v u_k'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2), \tag{4.49}$$

$$\Delta v_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) = \int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_v(t-t_2)} \beta_v u_k'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2), \tag{4.50}$$

$$\Delta\theta(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) = \int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_\theta(t-t_2)} \beta_\theta t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2). \tag{4.51}$$

The various tensors require equations for various correlations of u_i' and t', having the general form of $\langle b_i'(\boldsymbol{x}, t) b_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle$. These correlations represent the statistical properties of the fluid flow variables along the particle path and their prediction in the general flow situation remains a challenge (Minier 1999). In principle, the closed equations for these correlations can be obtained from (4.21)–(4.22) in the LHDI framework (Orszag 1968; Reeks 1992) by following a method similar to that implemented earlier to derive the equation for G. Here, we do not derive these equations and only present approximate expressions for these correlations, which are used later in this work and are accurate enough in homogeneous flows. These correlations $\langle b_i'(\boldsymbol{x}, t) b_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle$ are approximated by exponential functions with integral time scales $\tilde{T}_{b_i b_j} = 1/\beta_{b_i b_j}$,

$$\langle b_i'(\boldsymbol{x}, t) b_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle = \langle b_i'(\boldsymbol{x}, t) b_j'(\boldsymbol{x}, t)\rangle \exp(\beta_{b_i b_j}(t_2 - t)) \tag{4.52}$$

where summation is not implied for repeated indices i and j on the right-hand side of (4.52). For example,

$$\langle u_i'(\boldsymbol{x}, t) t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\rangle = \langle u_i'(\boldsymbol{x}, t) t'(\boldsymbol{x}, t)\rangle \exp(\beta_{u_i t}(t_2 - t)). \tag{4.53}$$

Before closing this section, we should mention that the uniform mean flow $\langle U_i \rangle$ and mean temperature $\langle T \rangle$ do not contribute to the various tensors and their effects appear in the equation for $\langle W \rangle$ only through the convective terms in (4.42) for the homogeneous case under consideration.

4.1.2. *Dispersed phase in inhomogeneous turbulent flows*

Now, we generalize the LHDI method, presented in the previous section for flows with uniform mean velocity and temperature, to inhomogeneous flows. The first step is to transform (3.2) for $W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)$ to a phase space $(\boldsymbol{y}, \boldsymbol{w}, \phi, t)$ in which $W(\boldsymbol{y}, \boldsymbol{w}, \phi, t)$ remains constant in the absence of turbulent fluctuations u_i' and t'. Consider the Lagrangian equations for the particle position z_i, velocity $\dot{z}_i$, and temperature ψ

without the fluctuating terms:

$$\frac{\mathrm{d}z_i}{\mathrm{d}s} = \dot{z}_i, \tag{4.54}$$

$$\frac{\mathrm{d}\dot{z}_i}{\mathrm{d}s} = \beta_v(\langle U_i\rangle - \dot{z}_i), \tag{4.55}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \beta_\theta(\langle T\rangle - \psi) + Q, \tag{4.56}$$

from which $(\boldsymbol{y}, \boldsymbol{w}, \phi)$ can be obtained and defined as

$$y_i = z_i(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s=0), \quad w_i = \dot{z}_i(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s=0), \quad \phi = \psi(\boldsymbol{x}, \boldsymbol{v}, \theta, t|s=0). \tag{4.57}$$

Here $\boldsymbol{y}$, $\boldsymbol{w}$, and ϕ are the position, velocity and temperature of the particle, respectively, at time $s = 0$ along the trajectory that is governed by (4.54)–(4.56) and passes through $(\boldsymbol{x}, \boldsymbol{v}, \theta)$ at time t. It should be noted that when $\langle U_i\rangle$ and $\langle T\rangle$ are constants and $Q = 0$, expressions for y_i, w_i and ϕ (as defined by (4.57)) are given by (4.6)–(4.8). In the transformed phase space, the equation for Green's function $\hat{G}$ is given by (4.9) with general definitions for operators l_i and M

$$l_i(\boldsymbol{y}, \boldsymbol{w}, \phi, t) = -\frac{\partial y_k}{\partial v_i}\frac{\partial}{\partial y_k} - \frac{\partial w_k}{\partial v_i}\frac{\partial}{\partial w_k} - \frac{\partial \phi}{\partial v_i}\frac{\partial}{\partial \phi}, \tag{4.58}$$

$$M(\boldsymbol{y}, \boldsymbol{w}, \phi, t) = -\frac{\partial y_k}{\partial \theta}\frac{\partial}{\partial y_k} - \frac{\partial w_k}{\partial \theta}\frac{\partial}{\partial w_k} - \frac{\partial \phi}{\partial \theta}\frac{\partial}{\partial \phi}. \tag{4.59}$$

Following a procedure identical to that described in the previous section, expressions for $\langle \beta_v u_i' W\rangle$ and $\langle \beta_\theta t' W\rangle$ can be obtained and are written in the original phase space $(\boldsymbol{x}, \boldsymbol{v}, \theta, t)$ as

$$\begin{aligned}
&-\langle \beta_v u_i' W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle \\
&\quad= \int_0^t \mathrm{d}t_2 \Bigg[\Bigg\langle \beta_v u_i'(\boldsymbol{x}, t)\left(G_{jk}(t_2|t)\frac{\partial}{\partial x_k} + \dot{G}_{jk}(t_2|t)\frac{\partial}{\partial v_k} + G_j(t_2|t)\frac{\partial}{\partial \theta}\right) \\
&\qquad \times \beta_v u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\Bigg\rangle \langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle + \Bigg\langle \beta_v u_i'(\boldsymbol{x}, t)\left(G^\theta(t_2|t)\frac{\partial}{\partial \theta}\right) \\
&\qquad \times \beta_\theta t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\Bigg\rangle \langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle\Bigg],
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
&-\langle \beta_\theta t' W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle \\
&\quad= \int_0^t \mathrm{d}t_2 \Bigg[\Bigg\langle \beta_\theta t'(\boldsymbol{x}, t)\left(G_{jk}(t_2|t)\frac{\partial}{\partial x_k} + \dot{G}_{jk}(t_2|t)\frac{\partial}{\partial v_k} + G_j(t_2|t)\frac{\partial}{\partial \theta}\right) \\
&\qquad \times \beta_v u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\Bigg\rangle \langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle + \Bigg\langle \beta_\theta t'(\boldsymbol{x}, t)\left(G^\theta(t_2|t)\frac{\partial}{\partial \theta}\right) \\
&\qquad \times \beta_\theta t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)\Bigg\rangle \langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t)\rangle\Bigg].
\end{aligned} \tag{4.61}$$

Here G_{jk}, $\dot{G}_{jk}$, G_j and G^θ are defined as

$$G_{jk}(t_2|t) = \frac{\partial}{\partial w_j} z_k(\boldsymbol{y}, \boldsymbol{w}, \phi, t_2|t) \equiv \frac{\partial x_k}{\partial v_j^{t_2-t}}, \tag{4.62}$$

$$\dot{G}_{jk}(t_2|t) = \frac{\mathrm{d}}{\mathrm{d}t} G_{jk}(t_2|t) \equiv \frac{\partial v_k}{\partial v_j^{t_2-t}}, \tag{4.63}$$

$$G_j(t_2|t) = \frac{\partial}{\partial w_j} \psi(\boldsymbol{y}, \boldsymbol{w}, \phi, t_2|t) \equiv \frac{\partial \theta}{\partial v_j^{t_2-t}}, \tag{4.64}$$

$$G^\theta(t_2|t) = \frac{\partial}{\partial \phi} \psi(\boldsymbol{y}, \boldsymbol{w}, \phi, t_2|t) \equiv \frac{\partial \theta}{\partial \theta^{t_2-t}}. \tag{4.65}$$

where the superscript $(t_2 - t)$ to any variable represents the value of that variable at time t_2.

The governing equations for G_{jk}, $\dot{G}_{jk}$, G_j and G^θ as obtained from the Lagrangian equations (4.54)–(4.56) are

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} G_{jk}(t_2|t) + \beta_v \frac{\mathrm{d}}{\mathrm{d}t} G_{jk} - \beta_v G_{ji} \frac{\partial \langle U_k \rangle}{\partial x_i} = \delta_{jk} \delta(t - t_2), \tag{4.66}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} G_j(t_2|t) - \beta_\theta G_{jk} \frac{\partial \langle T \rangle}{\partial x_k} - G_{jk} \frac{\partial Q}{\partial x_k} + \beta_\theta G_j = 0, \tag{4.67}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} G^\theta(t_2|t) + \beta_\theta G^\theta = \delta(t - t_2). \tag{4.68}$$

Equations (4.66) and (4.67) suggest that G_{jk}, $\dot{G}_{jk}$ and G_j depend only on the time variable for linear variation of Q, mean fluid velocity and temperature and when β_v and β_θ are constants. For such flows, expressions for the phase-space diffusion and heat currents as given by (4.60) and (4.61) can be written in the forms

$$\beta_v \langle u_i' W \rangle = -\left[\frac{\partial}{\partial x_k} \lambda_{ki} + \frac{\partial}{\partial v_k} \mu_{ki} + \frac{\partial}{\partial \theta} \omega_i - \gamma_i \right] \langle W \rangle, \tag{4.69}$$

$$\beta_\theta \langle t' W \rangle = -\left[\frac{\partial}{\partial x_k} \Lambda_k + \frac{\partial}{\partial v_k} \Pi_k + \frac{\partial}{\partial \theta} \Omega - \Gamma \right] \langle W \rangle. \tag{4.70}$$

Here, various tensors λ_{ki}, μ_{ki}, ω_i, γ_i, Λ_k, Π_k, Ω, and Γ are

$$\lambda_{ki} = \beta_v^2 \int_0^t \mathrm{d}t_2 \langle u_i'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle G_{jk}(t_2|t), \tag{4.71}$$

$$\mu_{ki} = \beta_v^2 \int_0^t \mathrm{d}t_2 \langle u_i'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle \frac{\mathrm{d}}{\mathrm{d}t} G_{jk}(t_2|t), \tag{4.72}$$

$$\omega_i = \beta_v^2 \int_0^t \mathrm{d}t_2 \langle u_i'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle G_j(t_2|t) + \beta_v \beta_\theta \int_0^t \mathrm{d}t_2 \langle u_i'(\boldsymbol{x}, t) t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle G^\theta(t_2|t), \tag{4.73}$$

$$\gamma_i = \beta_v^2 \int_0^t \mathrm{d}t_2 \left\langle \frac{\partial u_i'(\boldsymbol{x}, t)}{\partial x_k} u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \right\rangle G_{jk}(t_2|t), \tag{4.74}$$

$$\Lambda_k = \beta_v \beta_\theta \int_0^t \mathrm{d}t_2 \langle t'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle G_{jk}(t_2|t), \tag{4.75}$$

$$\Pi_k = \beta_v \beta_\theta \int_0^t \mathrm{d}t_2 \langle t'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle \frac{\mathrm{d}}{\mathrm{d}t} G_{jk}(t_2|t), \tag{4.76}$$

$$\Omega = \beta_v \beta_\theta \int_0^t \mathrm{d}t_2 \langle t'(\boldsymbol{x}, t) u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle G_j(t_2|t) + \beta_\theta^2 \int_0^t \mathrm{d}t_2 \langle t'(\boldsymbol{x}, t) t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle G^\theta(t_2|t), \tag{4.77}$$

$$\Gamma = \beta_v \beta_\theta \int_0^t \mathrm{d}t_2 \left\langle \frac{\partial t'(\boldsymbol{x}, t)}{\partial x_k} u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \right\rangle G_{jk}(t_2|t). \tag{4.78}$$

It should be noted that these tensors, except ω_i and Ω, can be written in the forms identical to those in (4.45)–(4.48), but with more general definitions of Δx_k, Δv_k and $\Delta\theta$:

$$\Delta x_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) = \int_0^t \mathrm{d}t_2 G_{jk}(t_2|t) \beta_v u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2), \tag{4.79}$$

$$\Delta v_k(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) = \int_0^t \mathrm{d}t_2 \dot{G}_{jk}(t_2|t) \beta_v u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2), \tag{4.80}$$

$$\Delta\theta(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) = \int_0^t \mathrm{d}t_2 G^\theta(t_2|t) \beta_\theta t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2). \tag{4.81}$$

The expressions for ω_i and Ω are

$$\omega_i = \langle \Delta\theta(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) \beta_v u_i'(\boldsymbol{x}, t) \rangle + \langle \Delta\theta_{u_j}(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) \beta_v u_i'(\boldsymbol{x}, t) \rangle, \tag{4.82}$$

$$\Omega = \langle \Delta\theta(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) \beta_\theta t'(\boldsymbol{x}, t) \rangle + \langle \Delta\theta_{u_j}(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0) \beta_\theta t'(\boldsymbol{x}, t) \rangle, \tag{4.83}$$

where $\Delta\theta_{u_j}(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)$, in the second term on the right-hand side of (4.82)–(4.83), represents the change in temperature of the particle due to $\beta_v u_j'(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2)$ along the trajectory of the particle starting at time zero and passing through $(\boldsymbol{x}, \boldsymbol{v}, \theta)$ at time t. This term $\Delta\theta_{u_j}(\boldsymbol{x}, \boldsymbol{v}, \theta, t|0)$ was equal to zero in the homogeneous case described in the previous section.

Before closing this section on LHDI, we should emphasize that LHDI generates terms containing λ_{ki} and γ_i in (4.69), and Λ_k and Γ in (4.70) which are responsible for certain new phenomena related to turbulent thermal diffusion and turbulent barodiffusion. Also, LHDI generates exact results for simple flow situations which are manifestations of RGT and ERGT. The new phenomena and compatibility of LHDI with ERGT are discussed in greater detail later in this paper. The DIA and closure represented by the classical Fokker–Planck equation failed to capture the terms containing λ_{ki} and γ_i and were not found compatible with RGT (Reeks 1991).

4.2. *Application of Van Kampen's method*

In this section, we use Van Kampen's method, the success of which in tackling the closure problem for isothermal two-phase flow has been established by Pozorski & Minier (1999). Van Kampen (1974*a*, *b*) originally proposed a cumulant expansion method for the solution of linear stochastic differential equations written for a vector process $\boldsymbol{Z}$:

$$\frac{\mathrm{d}\boldsymbol{Z}(t)}{\mathrm{d}t} = [A_0 + \alpha A_1(t)] \boldsymbol{Z}(t), \tag{4.84}$$

where the linear operators A_0 and A_1 are deterministic and stochastic in nature, respectively, and α is the level of fluctuations. A_0 is constant in time, and when the ensemble average of A_1 is time dependent, the proposed solution is

$$\frac{\mathrm{d}\langle \boldsymbol{Z}(t)\rangle}{\mathrm{d}t} = \left[A_0 + \alpha\langle A_1(t)\rangle + \alpha^2 \int_0^t \langle A_1'(t)\,\mathrm{e}^{sA_0} A_1'(t-s)\rangle\,\mathrm{e}^{-sA_0}\,\mathrm{d}s\right]\langle \boldsymbol{Z}(t)\rangle \tag{4.85}$$

for Kubo number $\alpha\tau_c \ll 1$. Here $A_1'(t) = A_1 - \langle A_1\rangle$ and τ_c is the autocorrelation time for A_1'. In the case of a single random oscillator problem where $\langle A_1(t)\rangle = 0$ and $A_1'(t)$ has Gaussian distribution, this approximate equation (4.85) yields the exact solution (Van Kampen 1974*a*). This method was then used by Van Kampen (1992) to solve the closure problems posed by the nonlinear stochastic equations, for N variables $\Theta_i, i = 1, 2, \ldots, N$, of the form

$$\frac{\mathrm{d}\Theta_i}{\mathrm{d}t} = K_i(\Theta_1, \Theta_2, \ldots, \Theta_N, S_n, t), \tag{4.86}$$

after transforming it to an equivalent linear equation for the phase-space density $W(\theta_i, t) = W(\theta_1, \theta_2, \ldots, \theta_N, t)$, written as

$$\frac{\partial}{\partial t} W(\theta_i, t) + \frac{\partial}{\partial \theta_i}[K_i(\theta_1, \theta_2, \ldots, \theta_N, S_n, t) W(\theta_i, t)] = 0. \tag{4.87}$$

Here, the K_i are functions of the Θ_i and some number M of stochastic variables S_n, $n = 1, 2, \ldots, M$, and the θ_i are phase-space variables corresponding to the Θ_i. Let K_i^0 and K_i^1 represent the stationary and stochastic parts of K_i, respectively. Then the comparison of (4.84) and (4.87) suggests that

$$A_0 = -\frac{\partial}{\partial \theta_i}(K_i^0), \quad A_1 = -\frac{\partial}{\partial \theta_i}(K_i^1). \tag{4.88}$$

Substituting A_0 and A_1 from (4.88) into (4.85) and replacing $\boldsymbol{Z}$ by W, yields

$$\frac{\partial\langle W\rangle}{\partial t} = -\frac{\partial}{\partial \theta_i}(K_i^0\langle W\rangle) + \frac{\partial}{\partial \theta_i}\int_0^t \mathrm{d}s \left\langle K_i^1(\boldsymbol{\theta}, t)\,\mathrm{e}^{sA_0}\frac{\partial}{\partial \theta_j} K_j^1(\boldsymbol{\theta}, t-s)\right\rangle \mathrm{e}^{-sA_0}\langle W\rangle \tag{4.89}$$

when $\langle K_i^1\rangle = 0$. For the phase-space density, the action of e^{sA_0} on any function $f(\boldsymbol{\theta})$ is given by (Van Kampen 1992; Pozorski & Minier 1999)

$$\mathrm{e}^{sA_0} f(\boldsymbol{\theta}) = f(\boldsymbol{\theta}^{-s})\frac{\mathrm{d}(\boldsymbol{\theta}^{-s})}{\mathrm{d}(\boldsymbol{\theta})}, \tag{4.90}$$

where $\boldsymbol{\theta}$ represents the value at time t and $\boldsymbol{\theta}^s$ represents the value at time $t + s$ along the trajectory which is determined from

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = K_i^0(\theta_1, \theta_2, \ldots, \theta_N), \tag{4.91}$$

and $\mathrm{d}(\boldsymbol{\theta}^{-s})/\mathrm{d}(\boldsymbol{\theta})$ stands for the Jacobian. Repeated application of (4.90) allows us to simplify (4.89) and the final general form of the closed transport equation for the probability density function is (Van Kampen 1992; Pozorski & Minier 1999)

$$\frac{\partial\langle W\rangle}{\partial t} + \frac{\partial}{\partial \theta_i}(K_i^0\langle W\rangle) = \frac{\partial}{\partial \theta_i}\int_0^t \mathrm{d}s \left\langle K_i^1(\boldsymbol{\theta}, t)\frac{\mathrm{d}(\boldsymbol{\theta}^{-s})}{\mathrm{d}(\boldsymbol{\theta})}\frac{\partial}{\partial \theta_j^{-s}} K_j^1(\boldsymbol{\theta}^{-s}, t-s)\right\rangle \frac{\mathrm{d}(\boldsymbol{\theta})}{\mathrm{d}(\boldsymbol{\theta}^{-s})}\langle W\rangle. \tag{4.92}$$

We now translate the present problem of a non-isothermal particle phase in turbulent flows which are governed by the Lagrangian equations (2.8)–(2.10). For these equations, we have

$$\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \\ \theta \end{bmatrix}, \quad \boldsymbol{K}^0 = \begin{bmatrix} \boldsymbol{v} \\ \beta_v(\langle \boldsymbol{U} \rangle - \boldsymbol{v}) \\ \beta_\theta(\langle T \rangle - \theta) + Q \end{bmatrix}, \quad \boldsymbol{K}^1 = \begin{bmatrix} \boldsymbol{0} \\ \beta_v \boldsymbol{u}' \\ \beta_\theta t' \end{bmatrix}, \tag{4.93}$$

and $\langle W \rangle = \langle W(\boldsymbol{x}, \boldsymbol{v}, \theta, t) \rangle$. Also, for any function $h = h[\boldsymbol{x}(\boldsymbol{x}^{-s}, \boldsymbol{v}^{-s}), \boldsymbol{v}(\boldsymbol{x}^{-s}, \boldsymbol{v}^{-s}), \theta(\boldsymbol{x}^{-s}, \boldsymbol{v}^{-s}, \theta^{-s})]$,

$$\frac{\partial h}{\partial v_j^{-s}} = \frac{\partial h}{\partial x_k}\frac{\partial x_k}{\partial v_j^{-s}} + \frac{\partial h}{\partial v_k}\frac{\partial v_k}{\partial v_j^{-s}} + \frac{\partial h}{\partial \theta}\frac{\partial \theta}{\partial v_j^{-s}}, \tag{4.94}$$

$$\frac{\partial h}{\partial \theta^{-s}} = \frac{\partial h}{\partial \theta}\frac{\partial \theta}{\partial \theta^{-s}}. \tag{4.95}$$

Equation (4.92) is now simplified by using (4.93)–(4.95) and is written

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_i}v_i + \frac{\partial}{\partial v_i}\beta_v(\langle U_i \rangle - v_i) + \frac{\partial}{\partial \theta}[\beta_\theta(\langle T \rangle - \theta) + Q]\right)\langle W(\boldsymbol{x}, \mathbf{v}, \theta, t)\rangle \\
&\quad = \frac{\partial}{\partial v_i}\left\{ \int_0^t \mathrm{d}s \frac{\mathrm{d}(\boldsymbol{\theta}^{-s})}{\mathrm{d}(\boldsymbol{\theta})}\left[\frac{\partial x_k}{\partial v_j^{-s}}\left\langle \beta_v u_i'(\boldsymbol{x}, t)\frac{\partial}{\partial x_k}\beta_v u_j'(\boldsymbol{\theta}^{-s}, t_2)\right\rangle\right.\right. \\
&\quad + \frac{\partial v_k}{\partial v_j^{-s}}\left\langle \beta_v u_i'(\boldsymbol{x}, t)\frac{\partial}{\partial v_k}\beta_v u_j'(\boldsymbol{\theta}^{-s}, t_2)\right\rangle + \frac{\partial \theta}{\partial v_j^{-s}}\left\langle \beta_v u_i'(\boldsymbol{x}, t)\frac{\partial}{\partial \theta}\beta_v u_j'(\boldsymbol{\theta}^{-s}, t_2)\right\rangle \\
&\quad \left.\left. + \frac{\partial \theta}{\partial \theta^{-s}}\left\langle \beta_v u_i'(\boldsymbol{x}, t)\frac{\partial}{\partial \theta}\beta_\theta t'(\boldsymbol{\theta}^{-s}, t_2)\right\rangle\right]\frac{\mathrm{d}(\boldsymbol{\theta})}{\mathrm{d}(\boldsymbol{\theta}^{-s})}\langle W \rangle\right\} \\
&\quad + \frac{\partial}{\partial \theta}\left\{ \int_0^t \mathrm{d}s \frac{\mathrm{d}(\boldsymbol{\theta}^{-s})}{\mathrm{d}(\boldsymbol{\theta})}\left[\frac{\partial x_k}{\partial v_j^{-s}}\left\langle \beta_\theta t'(\boldsymbol{x}, t)\frac{\partial}{\partial x_k}\beta_v u_j'(\boldsymbol{\theta}^{-s}, t_2)\right\rangle\right.\right. \\
&\quad + \frac{\partial v_k}{\partial v_j^{-s}}\left\langle \beta_\theta t'(\boldsymbol{x}, t)\frac{\partial}{\partial v_k}\beta_v u_j'(\boldsymbol{\theta}^{-s}, t_2)\right\rangle + \frac{\partial \theta}{\partial v_j^{-s}}\left\langle \beta_\theta t'(\boldsymbol{x}, t)\frac{\partial}{\partial \theta}\beta_v u_j'(\boldsymbol{\theta}^{-s}, t_2)\right\rangle \\
&\quad \left.\left. + \frac{\partial \theta}{\partial \theta^{-s}}\left\langle \beta_\theta t'(\boldsymbol{x}, t)\frac{\partial}{\partial \theta}\beta_\theta t'(\boldsymbol{\theta}^{-s}, t_2)\right\rangle\right]\frac{\mathrm{d}(\boldsymbol{\theta})}{\mathrm{d}(\boldsymbol{\theta}^{-s})}\langle W \rangle\right\},
\end{aligned} \tag{4.96}$$

where the argument $(\boldsymbol{\theta}^{-s}, t_2)$ is used to denote the argument $(\boldsymbol{x}^{-s}, \boldsymbol{v}^{-s}, \theta^{-s}, t-s)$. Now, changing variable s to $t - t_2$ and using (4.62)–(4.65) yields

$$\frac{\partial x_k}{\partial v_j^{-s}} = \frac{\partial x_k}{\partial v_j^{t_2 - t}} = G_{jk}(t_2|t), \tag{4.97}$$

$$\frac{\partial v_k}{\partial v_j^{-s}} = \frac{\partial v_k}{\partial v_j^{t_2 - t}} = \dot{G}_{jk}(t_2|t), \tag{4.98}$$

$$\frac{\partial \theta}{\partial v_j^{-s}} = \frac{\partial \theta}{\partial v_j^{t_2 - t}} = G_j(t_2|t), \tag{4.99}$$

$$\frac{\partial \theta}{\partial \theta^{-s}} = \frac{\partial \theta}{\partial \theta^{t_2 - t}} = G^\theta(t_2|t), \tag{4.100}$$

where $G_{jk}(t_2|t)$, $G_j(t_2|t)$, and $G^\theta(t_2|t)$ are governed by (4.66)–(4.68). In the case when the Jacobian is a function of only the time variable, changing variable s to $t - t_2$ on the right-hand side of (4.96) and using (4.97)–(4.100) gives expressions for $\beta_v\langle u_i' W\rangle$ and $\beta_\theta\langle t' W\rangle$ which are identical to the expressions obtained by LHDI as given by (4.69) and (4.70).

Also in this framework, for the case of homogeneous flow with constant values for $\langle U_i\rangle$ and $\langle T\rangle$, and $Q=0$,

$$x_i = x_i^{-s} + \frac{v_i^{-s}}{\beta_v}(1-\mathrm{e}^{-\beta_v s}) + \langle U_i\rangle s + \frac{\langle U_i\rangle}{\beta_v}(\mathrm{e}^{-\beta_v s}-1), \tag{4.101}$$

$$v_i = v_i^{-s}\mathrm{e}^{-\beta_v s} - \langle U_i\rangle(\mathrm{e}^{-\beta_v s}-1), \tag{4.102}$$

$$\theta = \theta^{-s}\mathrm{e}^{-\beta_\theta s} - \langle T\rangle(\mathrm{e}^{-\beta_\theta s}-1). \tag{4.103}$$

Using (4.101)–(4.103), equation (4.96) can be simplified and the resulting expression is identical to (4.42) that is obtained by the LHDI method for homogeneous flow.

4.3. *Extended random Galilean transformation invariance*

The kinetic equation for the particle phase in the case of isothermal two-phase flow is given by (3.6) and the closure problem posed by this equation is due to the presence of the unknown term $j_i = \beta_v\langle u_i' W(\boldsymbol{x},\boldsymbol{v},t)\rangle$, written in vector notation as $\boldsymbol{j} = \beta_v\langle \boldsymbol{u}' W(\boldsymbol{x},\boldsymbol{v},t)\rangle$. Reeks (1991) studied the isothermal case and showed that the usual closed form for $\beta_v\langle \boldsymbol{u}' W(\boldsymbol{x},\boldsymbol{v},t)\rangle = -\mu\partial\langle W\rangle/\partial\boldsymbol{v}$ as used in the classical Fokker–Planck equation does not preserve the symmetry of the random Galilean transformation (RGT), as proposed by Kraichnan (1965) in the context of the turbulence closure problem.

In RGT, a uniform translational velocity $\boldsymbol{u}_0$ is added to each realization of turbulent fluid flow, and when $\boldsymbol{u}_0$ has a Gaussian distribution over many realizations the statistical properties of this flow field are related to the statistical properties of the flow field without $\boldsymbol{u}_0$ and governed by some exact transformation rules. Any closure scheme is said to be compatible with RGT invariance if it satisfies the transformation rules. Using this RGT concept, Reeks (1991) obtained the general form for $\boldsymbol{j}$ and, later, Reeks (1992) formally derived the same equation by using Kraichnan's Lagrangian history direct interaction approximation (Kraichnan 1965).

Here, we propose an extended random Galilean transformation (ERGT) invariance, and further show that the kinetic equation (4.42) for a non-isothermal particle phase, as obtained by LHDI, is compatible with the constraint of ERGT. In the ERGT, we add a uniform translational velocity $\boldsymbol{u}_0$ and a uniform temperature θ_0 to each realization of fluid velocity and temperature, respectively. Both $\boldsymbol{u}_0$ and θ_0 have Gaussian distributions with zero mean and have finite correlation $\langle \boldsymbol{u}_0\theta_0\rangle$ over many realizations of the fluid velocity and temperature. The ensemble-average properties related to phase-space densities $W_G(\boldsymbol{x},\boldsymbol{v},\theta,t)$ and $W(\boldsymbol{x},\boldsymbol{v},\theta,t)$ are related by exact transformation rules and any closure scheme satisfying the rules is said to be compatible with ERGT. Here W_G is the phase-space density for particles in a new flow field with $\boldsymbol{u}_0$ and θ_0.

Now, we apply the ERGT concept to (2.8)–(2.10) with $Q=0$ and by considering β_v and β_θ to be constants. The addition of $\boldsymbol{u}_0$ and θ_0 to fluid velocity and temperature, respectively, changes the Lagrangian equations (2.8)–(2.10) with $Q=0$ to

$$\frac{\mathrm{d}\boldsymbol{X}_G}{\mathrm{d}t} = \boldsymbol{V}_G, \quad \frac{\mathrm{d}\boldsymbol{V}_G}{\mathrm{d}t} = \beta_v(\boldsymbol{U}_G-\boldsymbol{V}_G)+\beta_v\boldsymbol{u}_0, \quad \frac{\mathrm{d}T_{pG}}{\mathrm{d}t} = \beta_\theta(T_G-T_{pG})+\beta_\theta\theta_0, \tag{4.104}$$

where subscript G indicates the value of variables after adding $\boldsymbol{u}_0$ and θ_0. Under the conditions

$$\boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{U}_G(\boldsymbol{X}_G,t), \quad T(\boldsymbol{X},t) = T_G(\boldsymbol{X}_G,t), \tag{4.105}$$

$\boldsymbol{X}$, $\boldsymbol{V}$, T_p, and W are related to $\boldsymbol{X}_G$, $\boldsymbol{V}_G$, T_{pG}, and W_G by the following relations:

$$\boldsymbol{X} = \boldsymbol{X}_G - \boldsymbol{u}_0\int_0^t (1-\mathrm{e}^{-\beta_v s})\,\mathrm{d}s, \quad \boldsymbol{V} = \boldsymbol{V}_G - \boldsymbol{u}_0(1-\mathrm{e}^{-\beta_v t}), \quad T_p = T_{pG} - \theta_0(1-\mathrm{e}^{-\beta_\theta t}) \tag{4.106}$$

and

$$\begin{aligned} W_G(\boldsymbol{x},\boldsymbol{v},\theta,t) &= W\left(\boldsymbol{x}-\boldsymbol{u}_0\int_0^t (1-\mathrm{e}^{-\beta_v s})\,\mathrm{d}s, \boldsymbol{v}-\boldsymbol{u}_0(1-\mathrm{e}^{-\beta_v t}), \theta-\theta_0(1-\mathrm{e}^{-\beta_\theta t}), t\right) \\ &= \exp[\mathrm{i}\boldsymbol{u}_0\cdot\boldsymbol{k} + \mathrm{i}\theta_0 p]\,W(\boldsymbol{x},\boldsymbol{v},\theta,t), \end{aligned} \tag{4.107}$$

where

$$\boldsymbol{k} = \mathrm{i}\left[\int_0^t (1-\mathrm{e}^{-\beta_v s})\,\mathrm{d}s\frac{\partial}{\partial\boldsymbol{x}} + (1-\mathrm{e}^{-\beta_v t})\frac{\partial}{\partial\boldsymbol{v}}\right], \quad p = \mathrm{i}(1-\mathrm{e}^{-\beta_\theta t})\frac{\partial}{\partial\theta}, \quad \mathrm{i}^2 = -1. \tag{4.108}$$

The governing equation for $\langle W_G\rangle$ is

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x_i}v_i + \frac{\partial}{\partial v_i}\beta_v(\langle U_i\rangle - v_i) + \frac{\partial}{\partial\theta}\beta_\theta(\langle T\rangle-\theta)\right]\langle W_G\rangle \\ &\qquad = -\frac{\partial}{\partial v_i}\beta_v[\langle u_i' W_G\rangle + \langle u_{0i}W_G\rangle] - \frac{\partial}{\partial\theta}\beta_\theta[\langle t' W_G\rangle + \langle\theta_0 W_G\rangle], \end{aligned} \tag{4.109}$$

with the added unknown correlations $\langle u_{0i}W_G\rangle$ and $\langle\theta_0 W_G\rangle$ for which we now obtain exact expressions. Here u_{0i} is the ith component of velocity $\boldsymbol{u}_0$.

Equation (4.107) suggests that

$$\langle\boldsymbol{u}_0 W_G\rangle = -\mathrm{i}\frac{\partial}{\partial\boldsymbol{k}}\langle\exp[\mathrm{i}\boldsymbol{u}_0\cdot\boldsymbol{k}+\mathrm{i}\theta_0 p]\rangle\langle W(\boldsymbol{x},\boldsymbol{v},\theta,t)\rangle \tag{4.110}$$

and

$$\langle\theta_0 W_G\rangle = -\mathrm{i}\frac{\partial}{\partial p}\langle\exp[\mathrm{i}\boldsymbol{u}_0\cdot\boldsymbol{k}+\mathrm{i}\theta_0 p]\rangle\langle W(\boldsymbol{x},\boldsymbol{v},\theta,t)\rangle. \tag{4.111}$$

With an isotropic Gaussian distribution for $\boldsymbol{u}_0$ and θ_0,

$$\langle\exp[\mathrm{i}\boldsymbol{u}_0\cdot\boldsymbol{k}+\mathrm{i}\theta_0 p]\rangle = \exp[-\tfrac{1}{2}(\langle u_0^2\rangle\boldsymbol{k}\cdot\boldsymbol{k} + \langle\theta_0^2\rangle p^2) - p\boldsymbol{k}\cdot\langle\boldsymbol{u}_0\theta_0\rangle], \tag{4.112}$$

where $\langle u_0^2\rangle = \langle\boldsymbol{u}_0\cdot\boldsymbol{u}_0\rangle/3$. Substituting from (4.112) in (4.110) and (4.111) and using the relation (4.107), we obtain

$$\begin{aligned} \langle\boldsymbol{u}_0 W_G\rangle &= -\langle u_0^2\rangle\left[\int_0^t (1-\mathrm{e}^{-\beta_v s})\,\mathrm{d}s\frac{\partial}{\partial\boldsymbol{x}} + (1-\mathrm{e}^{-\beta_v t})\frac{\partial}{\partial\boldsymbol{v}}\right]\langle W_G\rangle \\ &\quad -\langle\theta_0\boldsymbol{u}_0\rangle(1-\mathrm{e}^{-\beta_\theta t})\frac{\partial\langle W_G\rangle}{\partial\theta}, \end{aligned} \tag{4.113}$$

or

$$\begin{aligned} \langle u_{0i} W_G\rangle &= -\langle u_0^2\rangle\left[\int_0^t (1-\mathrm{e}^{-\beta_v s})\,\mathrm{d}s\frac{\partial}{\partial x_i} + (1-\mathrm{e}^{-\beta_v t})\frac{\partial}{\partial v_i}\right]\langle W_G\rangle \\ &\quad -\langle\theta_0 u_{0i}\rangle(1-\mathrm{e}^{-\beta_\theta t})\frac{\partial\langle W_G\rangle}{\partial\theta}, \end{aligned} \tag{4.114}$$

and

$$\langle \theta_0 W_G\rangle = -\langle \theta_0 u_{0i}\rangle \left[\int_0^t (1-\mathrm{e}^{-\beta_v s})\,\mathrm{d}s \frac{\partial}{\partial x_i} + (1-\mathrm{e}^{-\beta_v t})\frac{\partial}{\partial v_i}\right]\langle W_G\rangle - \langle \theta_0^2\rangle (1-\mathrm{e}^{-\beta_\theta t})\frac{\partial \langle W_G\rangle}{\partial \theta}, \tag{4.115}$$

which are the *exact* closed expressions for $\langle u_{0i} W_G\rangle$ and $\langle \theta_0 W_G\rangle$.

Now we obtain the LHDI expressions for $\langle u_{0i} W_G\rangle$ and $\langle \theta_0 W_G\rangle$. The LHDI solution to closure problems of (4.109) can be obtained from (4.42) by substituting $u_i' + u_{0i}$, $t' + \theta_0$ and $\langle W_G\rangle$ in place of u_i', t' and $\langle W\rangle$, respectively, and using the condition that u_i' and t' are not correlated to u_{0i} and θ_0. The right-hand side of (4.42) and

$$\langle u_{0i}(\boldsymbol{x},t) u_{0j}(\boldsymbol{x},\boldsymbol{v},\theta,t|t_2)\rangle = \frac{\delta_{ij}}{3}\langle u_{0k}u_{0k}\rangle \equiv \delta_{ij}\langle u_0^2\rangle, \tag{4.116}$$

$$\langle u_{0i}(\boldsymbol{x},t)\theta_0(\boldsymbol{x},\boldsymbol{v},\theta,t|t_2)\rangle = \langle \theta_0(\boldsymbol{x},t) u_{0i}(\boldsymbol{x},\boldsymbol{v},\theta,t|t_2)\rangle = \langle \theta_0 u_{0i}\rangle, \tag{4.117}$$

$$\langle \theta_0(\boldsymbol{x},t)\theta_0(\boldsymbol{x},\boldsymbol{v},\theta,t|t_2)\rangle = \langle \theta_0^2\rangle, \tag{4.118}$$

give us

$$\begin{aligned}\langle u_{0i} W_G\rangle = &-\frac{\partial}{\partial x_i}\left\{\langle u_0^2\rangle\langle W_G\rangle \int_0^t \mathrm{d}t_2 (1-\mathrm{e}^{-\beta_v(t-t_2)})\right\} \\ &-\frac{\partial}{\partial v_i}\left\{\beta_v \langle u_0^2\rangle\langle W_G\rangle \int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_v(t-t_2)}\right\} \\ &-\frac{\partial}{\partial \theta}\left\{\beta_\theta \langle \theta_0 u_{0i}\rangle\langle W_G\rangle \int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_\theta(t-t_2)}\right\},\end{aligned} \tag{4.119}$$

$$\begin{aligned}\langle \theta_0 W_G\rangle = &-\frac{\partial}{\partial x_j}\left\{\langle \theta_0 u_{0j}\rangle\langle W_G\rangle \int_0^t \mathrm{d}t_2 (1-\mathrm{e}^{-\beta_v(t-t_2)})\right\} \\ &-\frac{\partial}{\partial v_j}\left\{\beta_v \langle \theta_0 u_{0j}\rangle\langle W_G\rangle \int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_v(t-t_2)}\right\} \\ &-\frac{\partial}{\partial \theta}\left\{\beta_\theta \langle \theta_0^2\rangle\langle W_G\rangle \int_0^t \mathrm{d}t_2\, \mathrm{e}^{-\beta_\theta(t-t_2)}\right\}.\end{aligned} \tag{4.120}$$

Since $\langle u_0^2\rangle$, $\langle \theta_0^2\rangle$ and $\langle \theta_0 u_{0i}\rangle$ are independent of $\boldsymbol{x}$, $\boldsymbol{v}$ and θ, a change of variable from $(t-t_2)$ to s in (4.119) and (4.120) results in expressions which are identical to the *exact* equations (4.114) and (4.115). This proves that the closure solutions as obtained by the LHDI method are compatible with the proposed extended random Galilean transformation.

5. Macroscopic (Eulerian) equations for particle-phase

The kinetic equation (3.3), along with the closed expressions given by (4.69) and (4.70), represents the closed transport equation for the probability density function in phase space $\boldsymbol{x}$, $\boldsymbol{v}$, θ and t. The numerical solution to this equation can be obtained by various methods, such as the Monte Carlo method, finite difference method, or the path integral method (Minier & Peirano 2001; Wehner & Wolfer 1987; Pandya & Mashayek 2001). The direct computation of the closed kinetic equation is computationally more expensive when there are many independent variables. Also, the computation becomes more difficult when the LHDI equations for various correlations of u_i' and t' appearing in (4.71)–(4.78), having the general form $\langle b_i'(\boldsymbol{x},t)b_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_2)\rangle$, are included. Another avenue, computationally less expensive, to extract useful information from the kinetic equation is to obtain macroscopic equations by taking various moments of the kinetic equation. These equations represent the transport equations for statistical quantities of interest for the particle phase in physical (Eulerian) space $(\boldsymbol{x},t)$ and are also known as the 'fluid' equations for the dispersed phase. These 'fluid' equations along with the algebraic expression of the form (4.52) for various correlations of u_i' and t' are easier to compute than the full kinetic equation, but contain less information.

The macroscopic equations governing the mean properties $N(\boldsymbol{x},t)$, $\overline{V}_i(\boldsymbol{x},t)$, and $\overline{\Theta}(\boldsymbol{x},t)$ for the particle phase are obtained by taking various moments of the probability density function equation (3.3) after substituting for $\langle u_i'W\rangle$ and $\langle t'W\rangle$ which are given by (4.69) and (4.70), respectively. Here, N, $\overline{V}_i$ and $\overline{\Theta}$ are the mean number density, the density-weighted ensemble average (referred to as the mean) of the velocity and temperature of the particle phase, respectively, at any location $\boldsymbol{x}$ at time t. These mean properties are defined as

$$N=\int\langle W\rangle\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\theta,\quad \overline{V}_j=\frac{1}{N}\int v_j\langle W\rangle\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\theta,\quad \overline{\Theta}=\frac{1}{N}\int\theta\langle W\rangle\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\theta, \tag{5.1}$$

and their transport equations are obtained from the kinetic equation, and written

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial x_i}[N\overline{V}_i]=0, \tag{5.2}$$

$$\frac{\partial\overline{V}_j}{\partial t}+\overline{V}_i\frac{\partial\overline{V}_j}{\partial x_i}+\frac{\partial\overline{v_i'v_j'}}{\partial x_i}=\beta_v(\langle U_j\rangle-\overline{V}_j)-[\overline{v_i'v_j'}+\bar{\lambda}_{ij}]\frac{\partial}{\partial x_i}\ln N-\frac{\partial}{\partial x_k}\bar{\lambda}_{kj}+\overline{\gamma}_j, \tag{5.3}$$

$$\frac{\partial\overline{\Theta}}{\partial t}+\overline{V}_i\frac{\partial\overline{\Theta}}{\partial x_i}+\frac{\partial\overline{v_i'\theta'}}{\partial x_i}=\beta_\theta(\langle T\rangle-\overline{\Theta})+Q-[\overline{v_i'\theta'}+\overline{\Lambda}_i]\frac{\partial}{\partial x_i}\ln N-\frac{\partial}{\partial x_i}\overline{\Lambda}_i+\overline{\Gamma}. \tag{5.4}$$

Here, $\bar{\lambda}_{kj}$, $\overline{\gamma}_j$, $\overline{\Lambda}_i$, and $\overline{\Gamma}$ are the density-weighted means of λ_{kj}, γ_j, Λ_i, and Γ, respectively. The density-weighted mean of any quantity, for example λ_{kj}, is defined as

$$\bar{\lambda}_{kj}=\frac{1}{N}\int\lambda_{kj}\langle W\rangle\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\theta, \tag{5.5}$$

and when λ_{kj} is independent of $\boldsymbol{v}$ and θ, $\bar{\lambda}_{kj}=\lambda_{kj}$. Reeks identified the term $\partial\overline{v_i'v_j'}/\partial x_i$ in (5.3) responsible for the transport of particles due to the change in turbulence intensity in space. This phenomenon, known as turbophoresis, was first recognized by Caporaloni *et al.* (1975) and formally derived by Reeks (1983) using LHDI in the p.d.f. approach. Also, Reeks (1992) referred to the term $\overline{\gamma}_j$ in (5.3) as the body force per unit mass of the dispersed phase, arising from inhomogeneities in the carrier fluid turbulence field. Recently, Pandya & Mashayek (2002*b*) have shown that the last two terms, $-(\partial/\partial x_k)\bar{\lambda}_{kj}+\overline{\gamma}_j$, present on the right-hand side of (5.3) are responsible for the two phenomena of turbulent thermal diffusion and turbulent barodiffusion in compressible fluid flow. These two phenomena were first recognized by Elperin, Kleeorin & Rogachevskii (1996, 1998). The effects of fluid compressibility on the particle phase are discussed further in detail in the next section.

Equation (5.4) can also be written

$$\overline{\Theta} = -\frac{1}{\beta_\theta}\frac{\partial \overline{v_i'\theta'}}{\partial x_i} + \langle T\rangle + \frac{Q}{\beta_\theta} - \frac{1}{\beta_\theta}[\overline{v_i'\theta'} + \overline{\Lambda_i}]\frac{\partial}{\partial x_i}\ln N - \frac{1}{\beta_\theta}\frac{\partial}{\partial x_i}\overline{\Lambda_i} + \frac{1}{\beta_\theta}\overline{\Gamma} - \frac{1}{\beta_\theta}\frac{\mathrm{D}\overline{\Theta}}{\mathrm{D}t}, \quad (5.6)$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \overline{V}_i\frac{\partial}{\partial x_i}. \quad (5.7)$$

The first term on the right-hand side of (5.6) is similar to the turbophoresis term. It is easy to see from (5.11) (see below) that $\overline{v_i'\theta'}$ is proportional to ω_i, which is written (see (4.73)) in terms of $\langle u_i'(\boldsymbol{x},t)u_j'(\boldsymbol{x},\boldsymbol{v},t|t_1)\rangle$ and $\langle u_i'(\boldsymbol{x},t)t'(\boldsymbol{x},\boldsymbol{v},t|t_1)\rangle$, and thus the spatial variations in fluid turbulent velocity correlation and turbulent heat flux contribute to heating or cooling of the particle phase. The term $(1/\beta_\theta)\overline{\Gamma}$ on the right-hand side of (5.6) represents the temperature source term arising from inhomogeneities in turbulent fluctuations of the fluid temperature field and is

$$\frac{1}{\beta_\theta}\overline{\Gamma} = \overline{\beta_v\int_0^t \mathrm{d}t_1\left\langle\frac{\partial t'(\boldsymbol{x},t)}{\partial x_k}u_j'(\boldsymbol{x},\boldsymbol{v},t|t_1)\right\rangle G_{jk}(\boldsymbol{x}_1,t_1;\boldsymbol{x},t)}, \quad (5.8)$$

where the overbar represents the density-weighted ensemble average. This term is analogous to $\overline{\gamma_j}$ in (5.3).

The macroscopic equations for the higher-order statistical properties

$$\overline{v_i'v_j'} = \frac{1}{N}\int(v_i-\overline{V}_i)(v_j-\overline{V}_j)\langle W\rangle\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\theta, \quad \overline{v_i'\theta'} = \frac{1}{N}\int(v_i-\overline{V}_i)(\theta-\overline{\Theta})\langle W\rangle\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\theta, \quad (5.9)$$

which are required to close the above set of equations (5.2)–(5.4), can be written

$$\frac{\partial\overline{v_j'v_n'}}{\partial t} + \overline{V}_i\frac{\partial}{\partial x_i}\overline{v_j'v_n'} + \frac{1}{N}\frac{\partial}{\partial x_i}[N\overline{v_i'v_j'v_n'}] = -\overline{v_i'v_j'}\frac{\partial\overline{V}_n}{\partial x_i} - \overline{v_i'v_n'}\frac{\partial\overline{V}_j}{\partial x_i} - 2\beta_v\overline{v_j'v_n'} - \overline{\lambda}_{kj}\frac{\partial\overline{V}_n}{\partial x_k} - \overline{\lambda}_{kn}\frac{\partial\overline{V}_j}{\partial x_k} + \overline{\mu}_{jn} + \overline{\mu}_{nj}, \quad (5.10)$$

$$\frac{\partial\overline{v_j'\theta'}}{\partial t} + \overline{V}_i\frac{\partial}{\partial x_i}\overline{v_j'\theta'} + \frac{1}{N}\frac{\partial}{\partial x_i}[N\overline{v_i'v_j'\theta'}] = -\beta_v\overline{v_j'\theta'} - \beta_\theta\overline{v_j'\theta'} - \overline{v_i'v_j'}\frac{\partial\overline{\Theta}}{\partial x_i} - \overline{v_i'\theta'}\frac{\partial\overline{V}_j}{\partial x_i} - \overline{\lambda}_{kj}\frac{\partial\overline{\Theta}}{\partial x_k} - \overline{\Lambda}_k\frac{\partial\overline{V}_j}{\partial x_k} + \overline{\omega}_j + \overline{\Pi}_j. \quad (5.11)$$

Also, the equation for the fluctuating temperature intensity of the particle phase,

$$\overline{\theta'\theta'} = \frac{1}{N}\int(\theta-\overline{\Theta})(\theta-\overline{\Theta})\langle W\rangle\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\theta, \quad (5.12)$$

can be written

$$\frac{\partial\overline{\theta'\theta'}}{\partial t} + \overline{V}_i\frac{\partial}{\partial x_i}\overline{\theta'\theta'} + \frac{1}{N}\frac{\partial}{\partial x_i}[N\overline{v_i'\theta'\theta'}] = -2\beta_\theta\overline{\theta'\theta'} - 2\overline{v_i'\theta'}\frac{\partial\overline{\Theta}}{\partial x_i} - 2\overline{\Lambda}_i\frac{\partial\overline{\Theta}}{\partial x_i} + 2\overline{\Omega}. \quad (5.13)$$

While deriving (5.10), (5.11) and (5.13), we have assumed the density-weighted averages of various tensors to be equal to their respective instantaneous values, e.g. $\overline{\lambda}_{ij} = \lambda_{ij}$. These equations, (5.10), (5.11) and (5.13), contain unknown third-order correlations $\overline{v_i'v_j'v_n'}$, $\overline{v_i'v_j'\theta'}$, and $\overline{v_i'\theta'\theta'}$. The equations for the third-order correlations would contain unknown fourth-order correlations and so on. Thus we are faced again with the well-known turbulence closure problem. To obtain the closed set of macroscopic equations at third order, the fourth-order correlations are written in terms of the second-order correlations by incorporating the quasi-normal condition. Incorporation of certain approximations, suggested by Zaichik (1999), can further simplify the equations for third-order correlations and result in algebraic expressions in terms of the second-order correlations. Another method of obtaining algebraic expressions for third-order correlations is based on the Chapman–Enskog method (Swailes *et al.* 1998; Derevich 2000). In this method, the kinetic or p.d.f. equation is solved analytically to obtain an approximate expression for the p.d.f. and then expressions for third-order correlations are derived from the known approximate expressions. We do not use this here to derive the transport equations or algebraic expressions for third-order correlations but leave it for future work.

Reeks (1993) studied the macroscopic equation for $\overline{v_i'v_j'}$ for the case of simple homogeneous shear flow and showed that the usual Boussinesq approximation for $\overline{v_i'v_j'}$ is not suitable. Instead, an algebraic expression, or constitutive relation, for $\overline{v_i'v_j'}$ is constituted of two parts: (i) a homogeneous component whose values are calculated as if the local fluid flow field were homogeneous and (ii) a deviatoric component involving terms proportional to the mean shearing of both phases. Now we obtain, in the simplest possible way, the algebraic expressions for $\overline{v_i'\theta'}$ and $\overline{\theta'\theta'}$ and show that these relations are composed of homogeneous and deviatoric components.

In the flow situations where the particle spatial density, shear stresses, $\overline{v_i'\theta}$ and $\overline{\theta'\theta'}$ at equilibrium are spatially uniform, neglecting the left-hand side of (5.11) and (5.13) allows us to write

$$\beta_v\overline{v_j'\theta'} + \beta_\theta\overline{v_j'\theta'} + \overline{v_i'\theta'}\frac{\partial \overline{V}_j}{\partial x_i} = -\overline{v_i'v_j'}\frac{\partial \overline{\Theta}}{\partial x_i} - \bar{\lambda}_{kj}\frac{\partial \overline{\Theta}}{\partial x_k} - \overline{\Lambda}_k\frac{\partial \overline{V}_j}{\partial x_k} + \bar{\varpi}_j + \overline{\Pi}_j, \tag{5.14}$$

and

$$\beta_\theta\overline{\theta'\theta'} = \left[-\overline{v_i'\theta'}\frac{\partial \overline{\Theta}}{\partial x_i} - \overline{\Lambda}_i\frac{\partial \overline{\Theta}}{\partial x_i}\right] + \overline{\Omega}. \tag{5.15}$$

The algebraic equation (5.14) can be further solved to obtain an explicit expression for $\overline{v_i'\theta'}$. Defining the tensor

$$\mathscr{A}_{ji} = (\beta_v + \beta_\theta)\delta_{ij} + \frac{\partial \overline{V}_j}{\partial x_i}, \tag{5.16}$$

and its inverse tensor as $\mathscr{A}_{nj}^{-1}$ such that $\mathscr{A}_{nj}^{-1}\mathscr{A}_{ji} = \delta_{ni}$, we obtain from (5.14)

$$\overline{v_n'\theta'} = \mathscr{A}_{nj}^{-1}\left[-\overline{v_i'v_j'}\frac{\partial \overline{\Theta}}{\partial x_i} - \bar{\lambda}_{kj}\frac{\partial \overline{\Theta}}{\partial x_k} - \Lambda_k\frac{\partial \overline{V}_j}{\partial x_k}\right] + \mathscr{A}_{nj}^{-1}\left[\bar{\varpi}_j + \overline{\Pi}_j\right]. \tag{5.17}$$

The term $\overline{\Omega}$ in (5.15) and the terms in the second bracket on the right-hand side of (5.17) are non-zero even when the mean flow of particle and fluid is uniform. Thus parts of these terms contain the homogeneous components and the remaining parts along with the first terms on the right-hand side of (5.15) and (5.17) constitute the deviatoric components.

The fluctuation correlations between fluid and particle flow variables can be obtained from the closed expressions for $\langle u_i' W\rangle$ and $\langle t' W\rangle$:

$$\beta_v \overline{u_i' v_j'} = -\bar{\lambda}_{ki} \frac{\partial \overline{V}_j}{\partial x_k} + \overline{\mu}_{ji}, \tag{5.18}$$

$$\beta_v \overline{u_i' \theta'} = -\bar{\lambda}_{ki} \frac{\partial \overline{\Theta}}{\partial x_k} + \bar{\varpi}_i, \tag{5.19}$$

$$\beta_\theta \overline{t' v_i'} = -\overline{\Lambda}_k \frac{\partial \overline{V}_i}{\partial x_k} + \overline{\Pi}_i, \tag{5.20}$$

$$\beta_\theta \overline{t' \theta'} = -\overline{\Lambda}_k \frac{\partial \overline{\Theta}}{\partial x_k} + \overline{\Omega}. \tag{5.21}$$

These correlations can be used to account for the 'back effects' of particles on the fluid turbulence (Zaichik 1999). Here we should point out, that similar to equations (5.15) and (5.17), part of the second terms on the right-hand side of these algebraic expressions (5.18)–(5.21) accounts for the homogeneous components and the remaining part along with the first terms on the right-hand side constitute the deviatoric components.

6. Effects of fluid compressibility

The closed kinetic equation and the macroscopic equations remain valid in the case of compressible fluid flows. The fluid compressibility affects the statistical properties of fluid velocity and temperature field which appear in these equations through various tensors. Also, the compressibility effects appear through the particle Reynolds number in the expressions for β_v and β_θ. In the case of particles dispersed in an ideal gas (with gas constant R and pressure P) obeying the equation of state $P = \rho R T$, compressibility leads to the phenomena of turbulent thermal diffusion and turbulent barodiffusion of particles. These phenomena for isothermal particles, which do not exchange heat with the surrounding fluid, were first described by Elperin *et al.* (1996, 1998) and recently identified in the macroscopic equation for the particle-phase mean velocity derived from the p.d.f. equation (Pandya & Mashayek 2002*b*).

We now discuss some more effects of fluid compressibility on a non-isothermal particle phase dispersed in ideal gas. Substitution for $\langle u_i' W\rangle$ and $\langle t' W\rangle$ from (4.69)–(4.70) in (3.3) results in a closed kinetic equation. In this resulting equation, the terms $(\partial/\partial v_i)(\gamma_i \langle W\rangle)$ and $(\partial/\partial\theta)(\Gamma\langle W\rangle)$ represent the convection in the phase space v_i and θ, respectively, arising due to the fluid turbulence fluctuations. Thus γ_i and Γ are parts of drift coefficients representing the 'drift velocities' of $\langle W\rangle$ along the phase-space variables v_i and θ, respectively. Before giving the approximate expressions for these drift velocities, we present approximate expressions for $\langle u_i'(\boldsymbol{x},t)\partial u_j'(\boldsymbol{x},t)/\partial x_j\rangle$, $\langle t'(\boldsymbol{x},t)\partial u_j'(\boldsymbol{x},t)/\partial x_j\rangle$, and G_{jk}. For low-Mach-number flows, the approximate expression

$$\partial u_i'/\partial x_i \approx -u_i' \partial \ln\langle\rho\rangle/\partial x_i \tag{6.1}$$

as obtained from the continuity equation (Elperin *et al.* 1998, 1997), and the ensemble average of equation of state ($\langle P\rangle \approx R\langle\rho\rangle\langle T\rangle$) allow us to write

$$\begin{aligned} \left\langle u_i'(\boldsymbol{x},t)\frac{\partial u_j'(\boldsymbol{x},t)}{\partial x_j}\right\rangle &\approx -\langle u_i'(\boldsymbol{x},t)u_j'(\boldsymbol{x},t)\rangle \frac{\partial \ln\langle\rho\rangle}{\partial x_j} \\ &\approx \langle u_i'(\boldsymbol{x},t)u_j'(\boldsymbol{x},t)\rangle \left[\frac{1}{\langle T\rangle}\frac{\partial\langle T(\boldsymbol{x},t)\rangle}{\partial x_j} - \frac{1}{\langle P\rangle}\frac{\partial\langle P(\boldsymbol{x},t)\rangle}{\partial x_j}\right] \end{aligned} \tag{6.2}$$

and

$$\left\langle t'(\boldsymbol{x},t)\frac{\partial u_j'(\boldsymbol{x},t)}{\partial x_j}\right\rangle \approx -\langle t'(\boldsymbol{x},t)u_j'(\boldsymbol{x},t)\rangle\frac{\partial \ln\langle\rho\rangle}{\partial x_j}$$

$$\approx \langle t'(\boldsymbol{x},t)u_j'(\boldsymbol{x},t)\rangle\left[\frac{1}{\langle T\rangle}\frac{\partial\langle T(\boldsymbol{x},t)\rangle}{\partial x_j}-\frac{1}{\langle P\rangle}\frac{\partial\langle P(\boldsymbol{x},t)\rangle}{\partial x_j}\right]. \quad (6.3)$$

Now we expand the solution of (4.66) around an isotropic case and write it in the form

$$G_{ji} = G_{ji}^0 + G_{ji}^1. \quad (6.4)$$

Here G_{ji}^0 represents the response function for the isotropic case and satisfies the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}G_{ji}^0 + \beta_v\frac{\mathrm{d}}{\mathrm{d}t}G_{ji}^0 = \delta_{ji}\delta(t-t'), \quad (6.5)$$

whereas G_{ji}^1 accounts for the effect of shear $\partial\beta_v\langle U_i\rangle/\partial x_k$ and satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}G_{ji}^1 + \beta_v\frac{\mathrm{d}}{\mathrm{d}t}G_{ji}^1 - G_{jk}^0\frac{\partial\beta_v\langle U_i\rangle}{\partial x_k} = 0. \quad (6.6)$$

Equation (6.5) with the initial conditions for G_{ij} and $\mathrm{d}G_{ij}/\mathrm{d}t$ equal to zero for $i \neq j$ suggests that

$$G_{ji}^0(t-t_1) = G^0(t-t_1)\delta_{ij} = \delta_{ij}(1-\mathrm{e}^{-\beta_v(t-t_1)})/\beta_v. \quad (6.7)$$

We write the expressions for γ_i and Γ as

$$\gamma_i = \beta_v^2\int_0^t \mathrm{d}t_1 G_{jk}(t_1|t)\frac{\partial}{\partial x_k}\langle u_i'(\boldsymbol{x},t)u_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle$$

$$-\beta_v^2\int_0^t \mathrm{d}t_1 G_{jk}(t_1|t)\left\langle\frac{\partial u_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)}{\partial x_k}u_i'(\boldsymbol{x},t)\right\rangle, \quad (6.8)$$

$$\Gamma = \beta_v\beta_\theta\int_0^t \mathrm{d}t_1 G_{jk}(t_1|t)\frac{\partial}{\partial x_k}\langle t'(\boldsymbol{x},t)u_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle$$

$$-\beta_v\beta_\theta\int_0^t \mathrm{d}t_1 G_{jk}(t_1|t)\left\langle\frac{\partial u_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)}{\partial x_k}t'(\boldsymbol{x},t)\right\rangle. \quad (6.9)$$

Now, the second term on the right-hand side of (6.8) and (6.9) can be further simplified by substituting for G_{ij} from (6.4) and (6.7), and introducing the usual exponential forms with integral time scales $\tilde{T}_L$ and $\tilde{T}_\theta$ for

$$\langle u_i'(\boldsymbol{x},t)u_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle = \langle u_i'(\boldsymbol{x},t)u_j'(\boldsymbol{x},t)\rangle\mathrm{e}^{-(t-t_1)/\tilde{T}_L}, \quad (6.10)$$

$$\langle t'(\boldsymbol{x},t)u_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle = \langle t'(\boldsymbol{x},t)u_j'(\boldsymbol{x},t)\rangle\mathrm{e}^{-(t-t_1)/\tilde{T}_\theta}. \quad (6.11)$$

The resulting expressions are

$$\gamma_i = \beta_v^2\int_0^t \mathrm{d}t_1 G_{jk}(t_1|t)\frac{\partial}{\partial x_k}\langle u_i'(\boldsymbol{x},t)u_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle$$

$$-\beta_v^2\int_0^t \mathrm{d}t_1 G_{jk}^1(t_1|t)\left\langle\frac{\partial u_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)}{\partial x_k}u_i'(\boldsymbol{x},t)\right\rangle$$

$$-\left\langle\frac{\partial u_k'(\boldsymbol{x},t)}{\partial x_k}u_i'(\boldsymbol{x},t)\right\rangle\beta_v\left\{\tilde{T}_L[1-\mathrm{e}^{(-t/\tilde{T}_L)}] + \frac{\tilde{T}_L}{\beta_v\tilde{T}_L+1}[\mathrm{e}^{-t(\beta_v+1/\tilde{T}_L)}-1]\right\}, \quad (6.12)$$

$$\Gamma = \beta_v\beta_\theta \int_0^t \mathrm{d}t_1 G_{jk}(t_1|t)\frac{\partial}{\partial x_k}\langle t'(\boldsymbol{x},t)u'_j(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle$$
$$-\beta_v\beta_\theta \int_0^t \mathrm{d}t_1 G^1_{jk}(t_1|t)\left\langle \frac{\partial u'_j(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)}{\partial x_k}t'(\boldsymbol{x},t)\right\rangle$$
$$-\beta_\theta\left\langle \frac{\partial u'_k(\boldsymbol{x},t)}{\partial x_k}t'(\boldsymbol{x},t)\right\rangle\left\{\tilde{T}_\theta[1-\mathrm{e}^{(-t/\tilde{T}_\theta)}]+\frac{\tilde{T}_\theta}{\beta_v\tilde{T}_\theta+1}[\mathrm{e}^{-t(\beta_v+1/\tilde{T}_\theta)}-1]\right\}. \tag{6.13}$$

Using these expressions for γ_i and Γ along with the expressions for λ_{kj} and Λ_k, the last two terms of (5.3) and (5.4) can be written

$$-\frac{\partial\lambda_{kj}}{\partial x_k}+\gamma_j = -\beta_v^2\int_0^t \mathrm{d}t_1\langle u'_j(\boldsymbol{x},t)u'_k(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle\frac{\partial G_{ki}(t_1|t)}{\partial x_i}$$
$$-\beta_v^2\int_0^t \mathrm{d}t_1 G^1_{ki}(t_1|t)\left\langle\frac{\partial u'_k(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)}{\partial x_i}u'_j(\boldsymbol{x},t)\right\rangle-\left\langle\frac{\partial u'_k(\boldsymbol{x},t)}{\partial x_k}u'_j(\boldsymbol{x},t)\right\rangle$$
$$\times\beta_v\left\{\tilde{T}_L\left[1-\mathrm{e}^{(-t/\tilde{T}_L)}\right]+\frac{\tilde{T}_L}{\beta_v\tilde{T}_L+1}\left[\mathrm{e}^{-t(\beta_v+1/\tilde{T}_L)}-1\right]\right\}, \tag{6.14}$$

$$-\frac{\partial\Lambda_k}{\partial x_k}+\Gamma = -\beta_v\beta_\theta\int_0^t \mathrm{d}t_1\langle t'(\boldsymbol{x},t)u'_j(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle\frac{\partial G_{jk}(t_1|t)}{\partial x_k}$$
$$-\beta_v\beta_\theta\int_0^t \mathrm{d}t_1 G^1_{jk}(t_1|t)\left\langle\frac{\partial u'_j(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)}{\partial x_k}t'(\boldsymbol{x},t)\right\rangle-\beta_\theta\left\langle\frac{\partial u'_k(\boldsymbol{x},t)}{\partial x_k}t'(\boldsymbol{x},t)\right\rangle$$
$$\times\left\{\tilde{T}_\theta\left[1-\mathrm{e}^{(-t/\tilde{T}_\theta)}\right]+\frac{\tilde{T}_\theta}{\beta_v\tilde{T}_\theta+1}\left[\mathrm{e}^{-t(\beta_v+1/\tilde{T}_\theta)}-1\right]\right\}. \tag{6.15}$$

The expressions for $\langle u'_i\partial u'_k/\partial x_k\rangle$ and $\langle t'\partial u'_k/\partial x_k\rangle$ for compressible gas have already been derived and are given by (6.2)–(6.3). Thus, the last term in (6.12)–(6.15) represents the turbulent thermal and pressure effects on the drift velocities of $\langle W\rangle$ along the phase-space variables v_i and θ, mean velocity $\overline{V}_j$, and mean temperature $\overline{\Theta}$. These effects, in the case of $\overline{V}_j$, are known as turbulent thermal diffusion and turbulent barodiffusion (Elperin *et al.* 1998; Pandya & Mashayek 2002*b*).

7. Model assessments

The macroscopic partial differential equations (5.2)–(5.4), (5.10)–(5.11) and (5.13) govern the evolution of statistical properties of the particle phase in Eulerian physical space and time. The cross-correlations between fluid and particle flow variables are governed by the algebraic equations (5.18)–(5.21). These equations are used here to predict the non-isothermal dispersed phase in homogeneous shear flow having constant values for mean velocity and temperature gradients. The predictions are compared with the results from our DNS study conducted in parallel with this modelling effort.

The details of the DNS will appear in Shotorban *et al.* (2002), and here we only provide a brief description. The simulations are carried out for an incompressible carrier phase with the energy equation decoupled from the continuity and momentum equations. Our numerical procedure for the carrier phase is similar to that adopted by Rogers, Moin & Reynolds (1986) who have considered the transport of a passive

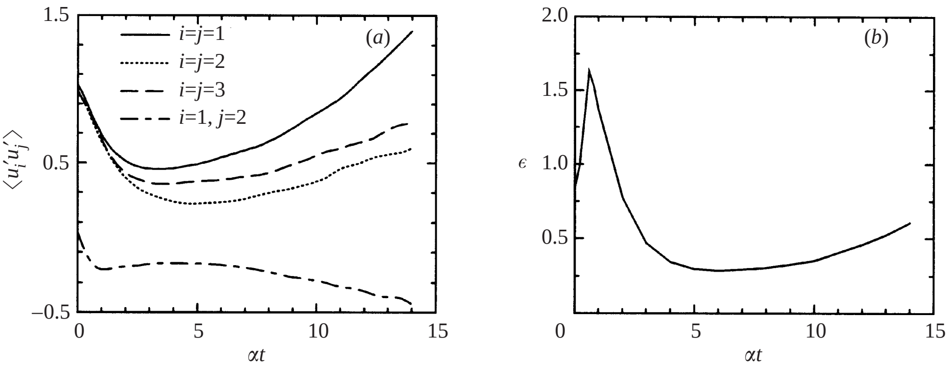

FIGURE 1. Temporal evolution of (a) Reynolds stresses $\langle u_i'u_j'\rangle$ and (b) dissipation rate ϵ, for the carrier phase.

scalar. Here, we treat the carrier-phase temperature as a passive scalar that has no effect on the evolution of the velocity field, but not the other way around. Only low mass loading ratios are considered so that the effects of the particles on the fluid can be neglected, i.e. one-way coupling.

The simulations are performed for a homogeneous shear flow, while a uniform mean temperature gradient is implemented in the energy equation. For simulating the carrier phase 128 × 128 × 128 grid points are used, with a uniform mean shear rate $\partial\langle U_1\rangle/\partial x_2 \equiv \alpha$, where from now onwards we consider streamwise, cross-stream and spanwise directions along the x_1-, x_2- and x_3- axes, respectively. For this velocity field, three different cases are considered for the mean temperature gradient. Case 1 refers to a constant mean temperature gradient imposed in the cross-stream direction, i.e. $\partial\langle T\rangle/\partial x_2$ is non-zero. Case 2 refers to a constant mean temperature gradient ($\partial\langle T\rangle/\partial x_3$) imposed in the spanwise direction. Case 3 refers to a constant mean temperature gradient ($\partial\langle T\rangle/\partial x_1$) imposed in the streamwise direction.

The particle phase is simulated by computing the Lagrangian trajectory, velocity and temperature, from equations (2.8)–(2.10), of a large number of particles ($N_p = 10^5$) after finding the carrier phase velocity and temperature at each time step from DNS. These particles are introduced in the flow domain at time $t = 0$ with velocity and temperature equal to the fluid velocity and temperature in their vicinity. The statistical properties of the particle phase are then calculated by averaging over the computed Lagrangian trajectories. It should be emphasized that in the DNS study all the variables are normalized by reference length (L_f), density (ρ_f), velocity (U_f) and temperature (T_f) scales. Thus the results presented in this section are in non-dimensional form; we do not change the present notation to represent the non-dimensional flow variables related to particle and fluid phases.

The predictions from macroscopic equations and algebraic equations require certain statistical properties of the carrier phase, which are presented in figures 1–3. Since the flow is homogeneous, the statistical properties depend only on time and the temporal evolution of the Reynolds stresses, $\langle u_i'u_j'\rangle$, and dissipation rate, ϵ, are shown in figure 1 for $\alpha = 2$. At time $t = 0$, the flow is isotropic and the shear stress $\langle u_1'u_2'\rangle$ evolves in time, from the initial value of zero, due to the imposed uniform mean shear rate. The temporal evolution of the temperature fluctuation intensity $\langle t't'\rangle$ for the three cases is shown in figure 2. In all of these cases, the mean temperature gradient is equal to 2. Another statistical property, namely, turbulent temperature flux $\langle u_i't'\rangle$, for the three

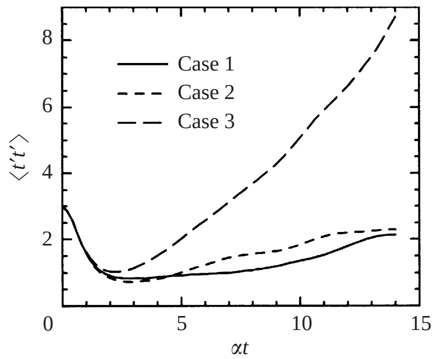

FIGURE 2. Temporal evolution of fluid temperature intensity $\langle t't' \rangle$ for three different cases.

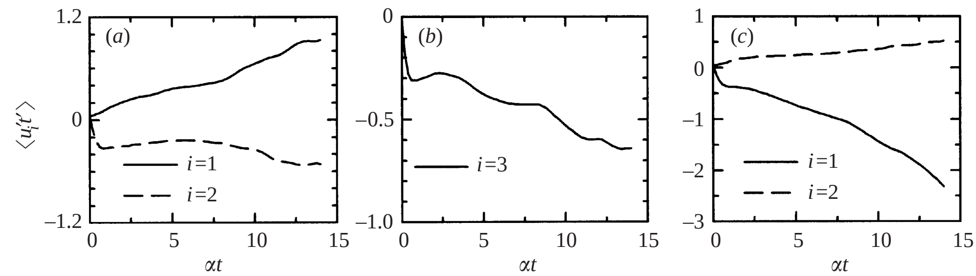

FIGURE 3. Temporal evolution of $\langle u_i't' \rangle$ for (a) case 1, (b) case 2, and (c) case 3.

cases is shown in figure 3. In cases 1 and 3, the flow and temperature fields have mirror symmetry about the plane (x_1, x_2) giving $\langle u_3't' \rangle = 0$ which is also observed in our DNS data. In case 2, we observe that $\langle u_1't' \rangle$ and $\langle u_2't' \rangle$ are equal to zero. These observations are consistent with the model proposed by Rogers, Mansour & Reynolds (1989) based on their DNS predictions, i.e.

$$\langle u_i't' \rangle = -D_{ij} \frac{\partial \langle T \rangle}{\partial x_j}, \tag{7.1}$$

where $D_{13} = D_{23} = D_{31} = D_{32} = 0$ for the present configuration of homogeneous shear flow. Only non-zero components of $\langle u_1't' \rangle$ are shown in figure 3.

The predictions also require prior knowledge of various integral time scales ($\tilde{T}_{b_i b_j} = 1/\beta_{b_i b_j}$). For the velocity field we take the usual approximation (Hyland *et al.* 1999*a*, *b*)

$$\tilde{T}_{u_i u_j} = 0.482 \frac{k}{\epsilon}. \tag{7.2}$$

Approximate values for the remaining time scales are obtained from the DNS data. For example, $\langle u_1'(\boldsymbol{x}, t) t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t | t_1) \rangle$ and $\langle u_1'(\boldsymbol{x}, t) t'(\boldsymbol{x}, t) \rangle$ are needed to calculate $\tilde{T}_{u_1 t}$. For the homogeneous shear flow, we take $\langle u_1'(\boldsymbol{x}, t) t'(\boldsymbol{x}, \boldsymbol{v}, \theta, t | t_1) \rangle$ equal to $\langle u_1'(t) t'(t_1) \rangle$ where $u_1'(t)$ and $t'(t_1)$ are fluid velocity and temperature fluctuations, as seen by the particle at times t and t_1, respectively, and the average is taken over all the particles. Since the present turbulent flow is unsteady, $\langle u_1'(t) t'(t_1) \rangle$ would depend on t_1. We take $t_1 = 1$, calculate $\langle u_1'(t) t'(t_1) \rangle / \langle u_1'(\boldsymbol{x}, t) t'(\boldsymbol{x}, t) \rangle$ for various $t \geqslant t_1$ and obtain an approximate value for $\tilde{T}_{u_1 t}$ such that $\exp(-(t - t_1)/\tilde{T}_{u_1 t})$ represents approximately the curve $\langle u_1'(t) t'(t_1) \rangle / \langle u_1'(\boldsymbol{x}, t) t'(\boldsymbol{x}, t) \rangle$ vs. $t - t_1$. The curve and the exponential function are shown in figure 4(*a*) for case 1. Also, the curves for $\langle t'(t) t'(t_1) \rangle / \langle t'(\boldsymbol{x}, t) t'(\boldsymbol{x}, t) \rangle$ vs.

Case	β_{tt}	β_{u1t}	β_{u2t}	β_{u3t}	β_{tu1}	β_{tu2}	β_{tu3}
1	1.9	1.6	2.35	–	2.0	1.6	–
2	1.9	–	–	2.35	–	–	1.6
3	1.9	2.0	2.35	–	1.3	1.0	–

TABLE 1. Values of $\beta_{b_ib_j}$.

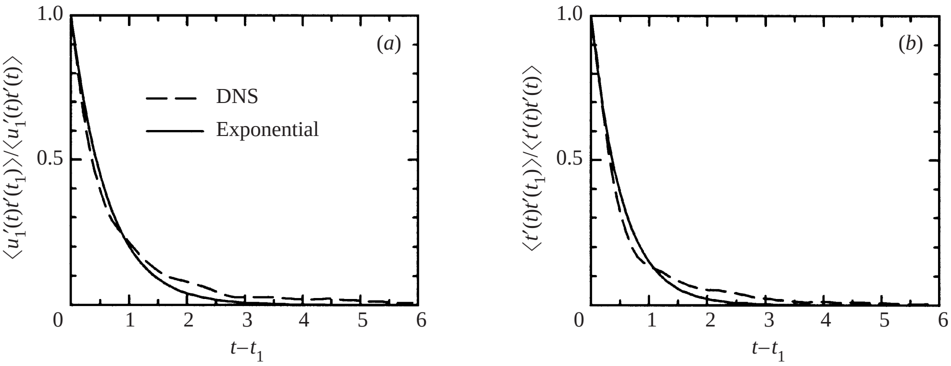

FIGURE 4. (a) Lagrangian velocity–temperature correlation function $\langle u_1'(t)t'(t_1)\rangle/\langle u_1'(\boldsymbol{x},t)t'(\boldsymbol{x},t)\rangle$ for case 1, (b) Lagrangian temperature–temperature correlation function $\langle t'(t)t'(t_1)\rangle/\langle t'(\boldsymbol{x},t)t'(\boldsymbol{x},t)\rangle$ for case 1.

$t-t_1$ as obtained from DNS data and the exponential form are shown in figure 4(b) for case 1. Though the value of $\tilde{T}_{u_1t}$ is a function of t_1 in this unsteady case, we take this value for $\tilde{T}_{u_1t}$ for all values of t_1 during the model predictions. The inverse of the required integral time constants ($\beta_{b_ib_j} = 1/\tilde{T}_{b_ib_j}$) obtained for different cases are shown in table 1. Recently, Pope (2002) has shown, for the self-similar state of homogeneous turbulent shear flow, that autocorrelations of fluid particle velocity can be made stationary by properly scaling the time. A similar approach can be followed to obtain, from DNS data, autocorrelations of fluid properties along the particle path and to assess its stationarity. Here we do not perform such calculation due to insufficient DNS data.

For the homogeneous shear flow, (5.2)–(5.3) give N = constant and $\overline{V_i} = \langle U_i\rangle$, which are also used for calculation of the particle phase in DNS. For cases 1 and 2, equation (5.4) is satisfied for $\overline{\Theta} = \langle T\rangle$ when $Q = 0$. For case 3, $\overline{\Theta} = \langle T\rangle$ is a solution in the presence of a source term $Q = \overline{V}_1\partial\overline{\Theta}/\partial x_1$. These results are consistent with the source term used in the present DNS which is designed to produce $\overline{\Theta} = \langle T\rangle$ and homogeneity.

In the case of homogeneous flow, macroscopic equation (5.10) reduces to

$$\frac{\mathrm{d}\overline{v_j'v_n'}}{\mathrm{d}t} = -\overline{v_i'v_j'}\frac{\partial\overline{V}_n}{\partial x_i} - \overline{v_i'v_n'}\frac{\partial\overline{V}_j}{\partial x_i} - 2\beta_v\overline{v_j'v_n'} - \bar{\lambda}_{kj}\frac{\partial\overline{V}_n}{\partial x_k} - \bar{\lambda}_{kn}\frac{\partial\overline{V}_j}{\partial x_k} + \bar{\mu}_{jn} + \bar{\mu}_{nj}. \tag{7.3}$$

These equations along with the initial conditions at $t = 0$ for $\overline{v_iv_j}$, and $\overline{V_i} = \langle U_i\rangle = \alpha\delta_{i1}x_2$ are numerically solved using a fourth-order-accurate Runge–Kutta method. The required initial conditions are given in table 2 and the analytical expressions for various tensors λ_{ij} and μ_{ij} are given in Appendix A. Also, the algebraic relation for $\overline{u_i'v_j'}$ given by (5.18) is computed using the analytical expressions for λ_{ij} and μ_{ij}.

τ_p	$\overline{v_1'v_1'}$	$\overline{v_2'v_2'}$	$\overline{v_3'v_3'}$	$\overline{v_1'v_2'}$
0.3	1.0246	0.9868	0.9833	2.861×10^{-02}
0.15	1.0249	0.9810	0.9917	2.629×10^{-02}
τ_p	$\overline{v_1'\theta'}$	$\overline{v_2'\theta'}$	$\overline{v_3'\theta'}$	$\overline{\theta'\theta'}$
0.3	3.465×10^{-2}	4.263×10^{-2}	-1.647×10^{-2}	3.0035

TABLE 2. Initial values for $\overline{v_i'v_j'}$, $\overline{v_i'\theta'}$ and $\overline{\theta'\theta'}$ at time $t = 0$.

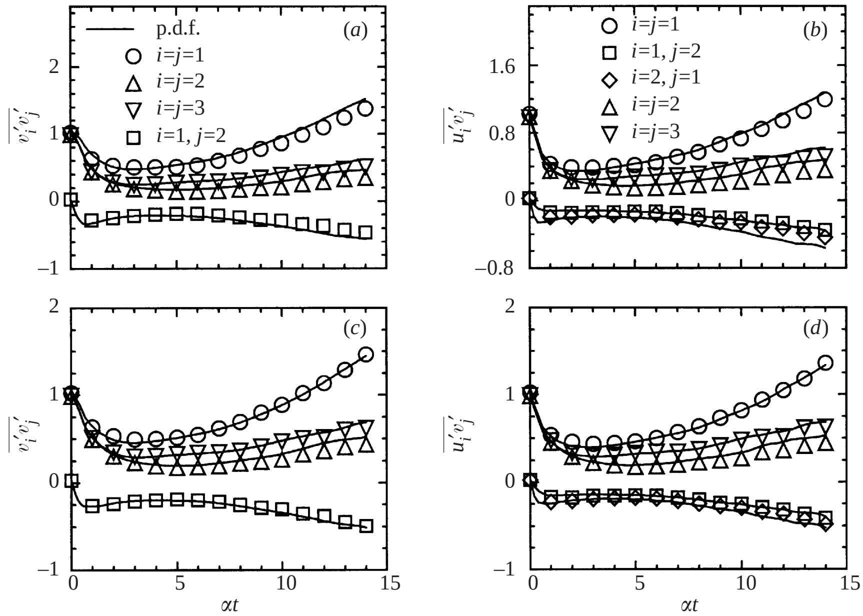

FIGURE 5. Temporal evolution of particle Reynolds stresses $\overline{v_i'v_j'}$ and fluid–particle velocity correlations $\overline{u_i'u_j'}$ (a,b) $\tau_p = 0.3$; (c,d) $\tau_p = 0.15$.

The temporal evolution of $\overline{v_i'v_j'}$ and $\overline{u_i'v_j'}$ for $\tau_p = 0.3$ is shown in figure 5(a,b), and for $\tau_p = 0.15$ in figure 5(c,d). The corresponding DNS data are also shown in the figure, and indicate encouraging agreements with the predicted results. The predictions appear more accurate for the particle with smaller time constant $\tau_p = 0.15$ than with $\tau_p = 0.3$. This could possibly be due to preferential concentration of particles (Mashayek 1998) with larger value of τ_p and resulting modifications in the statistics of fluid seen by particles. These modifications seem not to be captured properly by exponential form (4.52) with integral time scale given by (7.2). It should be noted that we have not shown certain components of tensors $\overline{v_i'v_j'}$ and $\overline{u_i'v_j'}$ as their values remain equal to zero. These components are $\overline{v_2'v_3'}$, $\overline{v_1'v_3'}$, $\overline{u_1'v_3'}$, $\overline{u_2'v_3'}$, $\overline{u_3'v_1'}$, and $\overline{u_3'v_2'}$.

The macroscopic equations (5.11) and (5.13) simplify to

$$\frac{\partial \overline{v_j'\theta'}}{\partial t} = -\beta_v \overline{v_j'\theta'} - \beta_\theta \overline{v_j'\theta'} - \overline{v_i'v_j'}\frac{\partial \overline{\Theta}}{\partial x_i} - \overline{v_i'\theta'}\frac{\partial \overline{V}_j}{\partial x_i} - \overline{\lambda}_{kj}\frac{\partial \overline{\Theta}}{\partial x_k} - \overline{\Lambda}_k \frac{\partial \overline{V}_j}{\partial x_k} + \bar{\varpi}_j + \overline{\Pi}_j, \quad (7.4)$$

and

$$\frac{\partial \overline{\theta'\theta'}}{\partial t} = -2\beta_\theta \overline{\theta'\theta'} - 2\overline{v_i'\theta'}\frac{\partial \overline{\Theta}}{\partial x_i} - 2\overline{\Lambda}_i \frac{\partial \overline{\Theta}}{\partial x_i} + 2\overline{\Omega}, \quad (7.5)$$

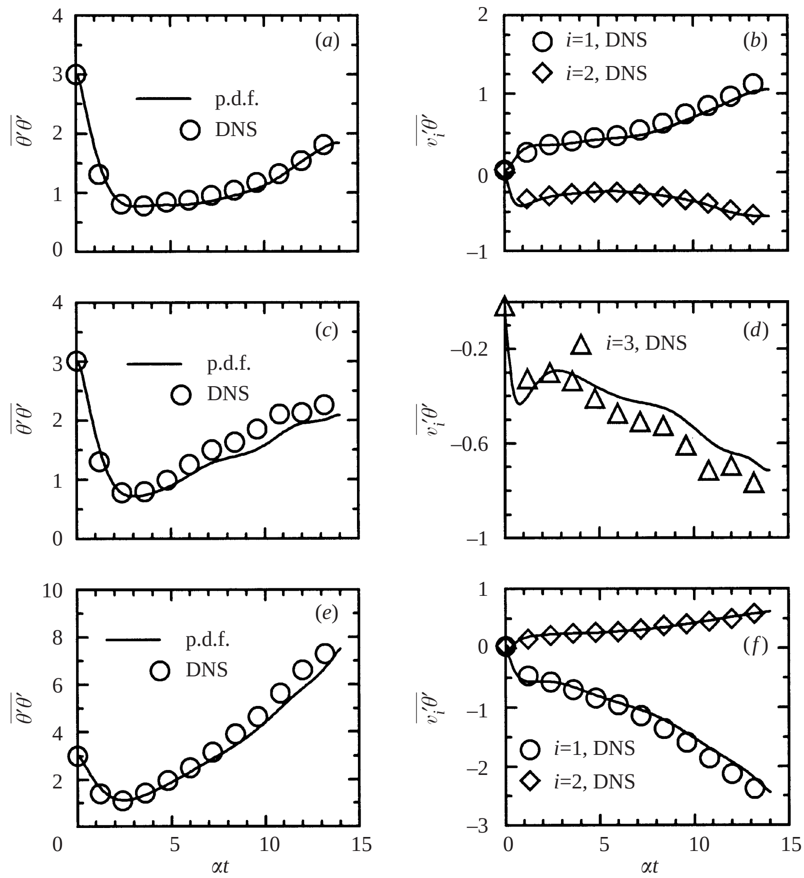

FIGURE 6. Temporal evolution of $\overline{\theta'\theta'}$ and $\overline{v_i'\theta'}$ for the three different cases: (a,b) case 1, (c,d) case 2, and (e,f) case 3.

in the case of homogeneous flows. These simplified equations (7.4)–(7.5) with initial conditions (see table 2) are solved, by a fourth-order-accurate Runge–Kutta method, using the analytical expressions for λ_{ki}, Λ_k, ω_i, Ω and Π_i (Appendix B) and the values for $\overline{v_i'v_j'}$ obtained from the predictions. These predictions for the three different cases are shown in figure 6 for particle time constant $\tau_p = 0.3$. Figure 6(a,b) shows the temporal evolution of $\overline{\theta'\theta'}$ and $\overline{v_i'\theta'}$ for case 1. Similarly, the temporal evolution of $\overline{\theta'\theta'}$ and $\overline{v_i'\theta'}$ for case 2 are shown in figure 6(c,d), and for case 3 in figure 6(e,f). The corresponding DNS data are also shown in the figure. In case 1, $\overline{v_3'\theta'} = 0$; in case 2, $\overline{v_1'\theta'} = \overline{v_2'\theta'} = 0$; and in case 3, $\overline{v_3'\theta'} = 0$. The predicted results are consistent with the DNS data and these zero values are not shown in figure 6. This figure suggests that the present predictions are in good agreement with the DNS data and capture the temporal behaviour of $\overline{\theta'\theta'}$ and $\overline{v_i'\theta'}$.

The algebraic relations (5.19)–(5.21) for fluid–particle cross-correlations are also computed and their temporal evolution along with the DNS data is shown in figure 7 for the three different cases. Figure 7(a,b) shows the predictions for case 1. Similarly figure 7(c,d) and figure 7(e,f) show the predictions for cases 2 and 3, respectively. Overall, the predictions are in good agreement with the DNS results. Again, those components of $\overline{v_i't'}$ and $\overline{u_i'\theta'}$ having values equal to zero, consistent with DNS data,

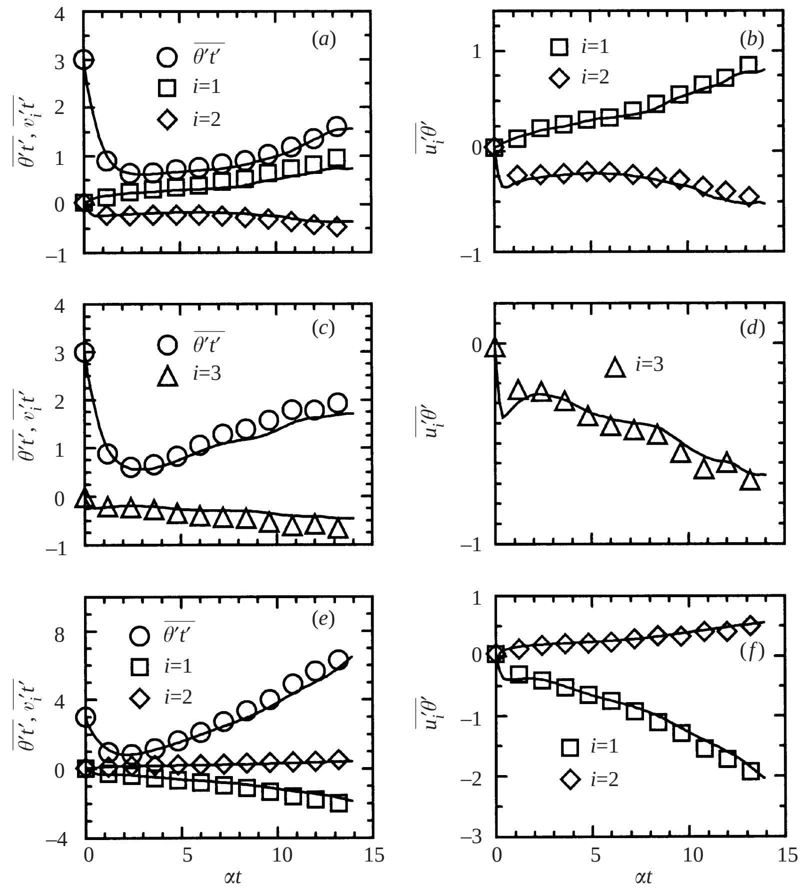

FIGURE 7. Temporal evolution of $\overline{\theta' t'}$, $\overline{v_i' t'}$ and $\overline{u_i' \theta'}$ for the three different cases: (a,b) case 1, (c,d) case 2, and (e,f) case 3. (Symbols for DNS data and solid lines for p.d.f. model predictions.)

are not shown in figure 7. Also, further comparisons (not shown here) indicated good agreements for particle statistical properties with DNS data for smaller particles at $\tau_p = 0.15$.

8. Concluding remarks

A common situation of two-phase non-isothermal turbulent flow has been considered in an attempt to derive 'fluid-like' equations, in a Eulerian framework, for statistical properties of the particle phase from the first-principle Lagrangian equations for such a situation. Of the two well-known available approaches (RANS and p.d.f.) whose end results are Eulerian equations, the single-point p.d.f. approach has been considered for the description of the non-isothermal particle phase with the known fluid flow properties as external variables. This approach is more suitable for deriving the boundary conditions for the particle phase (Alipchenkov *et al.* 2001) and the transition from the Lagrangian to Eulerian framework occurs in a natural manner without an artifact. Certain closure problems have appeared at various stages during the derivation of the particle Eulerian equations. The closure problems appeared first, in the form of unknown correlations $\langle \boldsymbol{u}' W \rangle$ and $\langle t' W \rangle$, in the equation for the ensemble average of phase-space density $\langle W \rangle$ which is known as the kinetic or p.d.f. equation. These problems have been solved with both Kraichnan's LHDI approach and Van Kampen's method, which resulted in identical closed kinetic or p.d.f. equations, also shown to be compatible with the newly proposed transformation constraint of extended random Galilean transformation. These closed solutions, for $\langle \boldsymbol{u}' W \rangle$ and $\langle t' W \rangle$, then raised another closure problem requiring the statistical properties of the fluid flow variables as seen by the particles, in the form $\langle b'_i(\boldsymbol{x}, t) b'_j(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle$. In principle, the governing equations for these correlations can be obtained in the LHDI framework. Instead of providing the LHDI solution, exponential forms for these properties have been assumed which allow us to obtain analytical expressions for various tensors in the simple situation of homogeneous shear flow with uniform mean temperature gradients.

In the final step of the transition from a Lagrangian to Eulerian framework, various moments of the closed kinetic equations have been taken to derive the required Eulerian equations. These equations were found to contain closure problems due to the appearance of higher-order correlations in the equations for lower-order ones for the case of non-homogeneous flows. Only the available methods in the framework of the kinetic approach have been discussed to solve such problems. The effects of the fluid compressibility on the particle phase have been discussed using the particle Eulerian equations and two new phenomena of turbulent thermal diffusion and turbulent barodiffusion of the particle phase along with other similar new phenomena related to particle temperature have been quantified. In the case of homogeneous flows with uniform mean gradients for fluid velocity and temperature, the Eulerian equation predictions have been compared with direct numerical simulations data. These comparisons indicate the success, in homogeneous flows, of the prescription of the kinetic approach leading to Eulerian equations presented in this paper. This leaves us with the unfinished important task of assessment in non-homogeneous situation after properly tackling the closure problems posed by the correlations $\langle b'_i(\boldsymbol{x}, t) b'_j(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle$ and higher-order correlations appearing in the particle Eulerian equations. These will be considered in future work.

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Appendix A. Analytical expressions for velocity-related tensors

In this appendix, we present analytical expressions for various tensors λ_{ij} and μ_{ij} which are required to predict $\overline{v'_j v'_n}$ and $\overline{u'_i v'_j}$ from (7.3) and (5.18) for homogeneous flow with a uniform mean velocity gradient. In general, these tensors are functions of $\boldsymbol{x}$, $\boldsymbol{v}$ and t. In the present situations, analytical expressions can be obtained when an exponential form is assumed for various correlations of the type $\langle b'_i(\boldsymbol{x}, t) b'_j(\boldsymbol{x}, \boldsymbol{v}, \theta, t|t_2) \rangle$.

The governing equation (4.66) suggests that $G_{jk}(t_2|t)$ depends only on the time variable for uniform $\partial \langle U_k \rangle / \partial x_i$ and constant values for β_v. Also, G_{jk} represents the displacement of the particle in the k-direction resulting from an impulsive force $\delta(t - t_2)$ applied in the j-direction (Reeks 1992). Here, we use a more general equation for G_{jk} which includes a term accounting for the initial correlation between particle velocity and fluid velocity. The equation is written (Hyland *et al.* 1999*a, b*)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}G_{jk}(t_2|t)+\beta_v\frac{\mathrm{d}}{\mathrm{d}t}G_{jk}-\beta_v G_{ji}\frac{\partial\langle U_k\rangle}{\partial x_i}=\delta_{jk}\delta(t-t_2)+A_{jk}\delta(t), \tag{A 1}$$

where A_{jk} is given by

$$A_{jk}(t_2)=\frac{a_v}{\beta_v}\delta_{jk}\delta(t_2) \tag{A 2}$$

when $V_i=a_vU_i$ at time $t=0$. For stationary turbulence we have $G_{ij}(t_2|t)=G_{ij}(t-t_2)$ and $G_{ij}(t_1=0|t)$ is then given by (Hyland *et al.* 1999*a, b*)

$$G_{ij}(t_1=0|t)=\left[1+\frac{a_v}{\beta_v}\delta(t_1=0)\right]\left\{\frac{\delta_{ij}}{\beta_v}(1-\mathrm{e}^{-\beta_v t})+\delta_{i2}\delta_{j1}\frac{\alpha}{\beta_v^2}\right.$$
$$\left.\times[2(\mathrm{e}^{-\beta_v t}-1)+\beta_v t(1+\mathrm{e}^{-\beta_v t})]\right\}, \tag{A 3}$$

where $\alpha=\partial\langle U_1\rangle/\partial x_2$. We approximate various correlations for fluctuating fluid variables, in the general form of $\langle b_i'(\boldsymbol{x},t)b_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle$, using the integral time scale $\tilde{T}_{b_ib_j}=1/\beta_{b_ib_j}$, as

$$\langle b_i'(\boldsymbol{x},t)b_j'(\boldsymbol{x},\boldsymbol{v},\theta,t|t_1)\rangle=\langle b_i'(\boldsymbol{x},t)b_j'(\boldsymbol{x},t)\rangle\exp(-\beta_{b_ib_j}(t-t_1)) \tag{A 4}$$

and consider them stationary, i.e. $\beta_{b_ib_j}$ are constants and the correlations are independent of t_1 and only depend on the difference $(t-t_1)$. Here summation is not implied for repeated indices i and j on the right-hand side of (A 4). Using these expressions for G_{ij} and fluid correlations, expressions (4.71)–(4.72) for tensors λ_{ki} and μ_{ki} are simplified and given by

$$\lambda_{ki}=\beta_v^2\langle u_i'(\boldsymbol{x},t)u_k'(\boldsymbol{x},t)\rangle I_1(\beta_{u_iu_k})+\beta_v^2\langle u_i'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle\delta_{k1}I_2(\beta_{u_iu_2}) \tag{A 5}$$

and

$$\mu_{ki}=\beta_v^2\langle u_i'(\boldsymbol{x},t)u_k'(\boldsymbol{x},t)\rangle J_1(\beta_{u_iu_k})+\beta_v^2\langle u_i'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle\delta_{k1}J_2(\beta_{u_iu_2}). \tag{A 6}$$

Here, the functions I_1, I_2, J_1 and J_2 are given by

$$I_1(\beta)=\int_0^t \mathrm{e}^{-\beta(t-t_1)}G_{11}(t-t_1)\,\mathrm{d}t_1$$
$$=\frac{a_v}{\beta_v^2}\,\mathrm{e}^{-\beta t}(1-\mathrm{e}^{-\beta_v t})+\frac{\mathrm{e}^{-(\beta_v+\beta)t}-1}{\beta_v(\beta_v+\beta)}-\frac{\mathrm{e}^{-\beta t}-1}{\beta_v\beta}, \tag{A 7}$$

$$I_2(\beta)=\int_0^t \mathrm{e}^{-\beta(t-t_1)}G_{21}(t-t_1)\,\mathrm{d}t_1$$
$$=\frac{a_v\alpha}{\beta_v^3}\mathrm{e}^{-\beta t}[2(\mathrm{e}^{-\beta_v t}-1)+\beta_v t(1+\mathrm{e}^{-\beta_v t})]+\frac{\alpha}{\beta_v^2}\left\{2\left[\frac{1-\mathrm{e}^{-(\beta_v+\beta)t}}{\beta_v+\beta}+\frac{\mathrm{e}^{-\beta t}-1}{\beta}\right]\right.$$
$$\left.+\frac{\beta_v}{\beta^2}[1-\mathrm{e}^{-\beta t}(1+\beta t)]+\frac{\beta_v}{(\beta_v+\beta)^2}[1-\mathrm{e}^{-(\beta_v+\beta)t}(1+\beta_v t+\beta t)]\right\}, \tag{A 8}$$

$$J_1(\beta)=\int_0^t \mathrm{e}^{-\beta(t-t_1)}\frac{\mathrm{d}}{\mathrm{d}t}G_{11}(t-t_1)\,\mathrm{d}t_1$$
$$=\frac{a_v}{\beta_v}\,\mathrm{e}^{-(\beta_v+\beta)t}+\frac{1-\mathrm{e}^{-(\beta_v+\beta)t}}{\beta_v+\beta}, \tag{A 9}$$

and

$$J_2(\beta) = \int_0^t e^{-\beta(t-t_1)} \frac{d}{dt} G_{21}(t-t_1)\, dt_1$$
$$= \frac{a_v \alpha}{\beta_v^2} e^{-\beta t}[1 - e^{-\beta_v t}(1+\beta_v t)] + \frac{\alpha}{\beta_v}\left\{\frac{1-e^{-\beta t}}{\beta} + \frac{e^{-(\beta_v+\beta)t}-1}{\beta_v+\beta}\right.$$
$$\left. - \frac{\beta_v}{(\beta_v+\beta)^2}[1 - e^{-(\beta_v+\beta)t}(1+\beta_v t + \beta t)]\right\}. \tag{A 10}$$

Appendix B. Analytical expressions for temperature-related tensors

The tensors Λ_k and Π_k depend on G_{jk} and can be obtained from (4.75) and (4.76) as

$$\Lambda_k = \beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_k'(\boldsymbol{x},t)\rangle I_1(\beta_{tu_k}) + \beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle\delta_{k1}I_2(\beta_{tu_2}) \tag{B 1}$$

and

$$\Pi_k = \beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_k'(\boldsymbol{x},t)\rangle J_1(\beta_{tu_k}) + \beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle\delta_{k1}J_2(\beta_{tu_2}), \tag{B 2}$$

where I_1, I_2, J_1 and J_2 are as given in Appendix A.

Following the procedure given by Hyland *et al.* (1999*a*), equation (4.68) for G^θ is modified to properly account for the initial correlation between particle temperature and fluid temperature at time $t = 0$, written

$$\frac{d}{dt}G^\theta(t_2|t) + \beta_\theta G^\theta = \delta(t-t_2) + C_\theta\delta(t). \tag{B 3}$$

Here $C_\theta\delta(t)$ accounts for the initial correlation and

$$C_\theta = \frac{a_\theta}{\beta_\theta}\delta(t_2) \tag{B 4}$$

with $T_p = a_\theta T$ at time $t = 0$. The solution of (B 3) is

$$G^\theta(t_1|t) = e^{-\beta_\theta(t-t_1)} + \frac{a_\theta}{\beta_\theta}\delta(t_1)\, e^{-\beta_\theta t}. \tag{B 5}$$

Using $G^\theta(t_1|t)$, the solution of equation (4.67) for G_i, along with the assumed exponential form for various correlations of fluid flow variables, the expressions for ω_i and Ω as given by (4.73) and (4.77) are simplified for the different cases.

Case 1: When $\partial\langle T\rangle/\partial x_2 \neq 0$ and $\partial\langle T\rangle/\partial x_1 = \partial\langle T\rangle/\partial x_3 = 0$.
Equation (4.67) for G_i gives $G_1 = G_3 = 0$ and

$$G_2(t_1|t) = \int_{t_1}^t e^{-\beta_\theta(t-s)}\beta_\theta G_{22}(t_1|s)\frac{\partial\langle T\rangle}{\partial x_2}\, ds. \tag{B 6}$$

Now Ω and ω_i are given by

$$\Omega = \beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle K_1(\beta_{tu_2})\frac{\partial\langle T\rangle}{\partial x_2} + \beta_\theta^2\langle t'(\boldsymbol{x},t)t'(\boldsymbol{x},t)\rangle K_2(\beta_{tt}) \tag{B 7}$$

and

$$\omega_i = \beta_v^2\langle u_i'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle K_1(\beta_{u_iu_2})\frac{\partial\langle T\rangle}{\partial x_2} + \beta_v\beta_\theta\langle u_i'(\boldsymbol{x},t)t'(\boldsymbol{x},t)\rangle K_2(\beta_{u_it}), \tag{B 8}$$

where

$$K_1(\beta) = \frac{1}{\partial\langle T\rangle/\partial x_2}\int_0^t \mathrm{e}^{-\beta(t-t_1)}G_2(t-t_1)\,\mathrm{d}t_1$$
$$= \frac{1}{\beta_v(\beta_v-\beta_\theta)}\left[\frac{a_v}{\beta_v}\,\mathrm{e}^{-\beta t}(\beta_v-\beta_\theta-\beta_v\,\mathrm{e}^{-\beta_\theta t}+\beta_\theta\,\mathrm{e}^{-\beta_v t})+\frac{\beta_\theta-\beta_v}{\beta}(\mathrm{e}^{-\beta t}-1)\right.$$
$$\left.+\frac{\beta_v}{\beta_\theta+\beta}(\mathrm{e}^{-(\beta_\theta+\beta)t}-1)-\frac{\beta_\theta}{\beta_v+\beta}(\mathrm{e}^{-(\beta_v+\beta)t}-1)\right] \tag{B 9}$$

and

$$K_2(\beta) = \int_0^t \mathrm{e}^{-\beta(t-t_1)}G^\theta(t-t_1)\,\mathrm{d}t_1$$
$$= \frac{1}{\beta_\theta(\beta_\theta+\beta)}\{[a_\theta(\beta+\beta_\theta)-\beta_\theta]\,\mathrm{e}^{-(\beta+\beta_\theta)t}+\beta_\theta\}. \tag{B 10}$$

Case 2: When $\partial\langle T\rangle/\partial x_3 \neq 0$ and $\partial\langle T\rangle/\partial x_1 = \partial\langle T\rangle/\partial x_2 = 0$.
For this case, $G_1 = G_2 = 0$ and

$$G_3(t_1|t) = \int_{t_1}^t \mathrm{e}^{-\beta_\theta(t-s)}\beta_\theta G_{33}(t_1|s)\frac{\partial\langle T\rangle}{\partial x_3}\,\mathrm{d}s. \tag{B 11}$$

Also, Ω and ω_i are given by

$$\Omega = \beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_3'(\boldsymbol{x},t)\rangle K_1(\beta_{tu_3})\frac{\partial\langle T\rangle}{\partial x_3}+\beta_\theta^2\langle t'(\boldsymbol{x},t)t'(\boldsymbol{x},t)\rangle K_2(\beta_{tt}) \tag{B 12}$$

and

$$\omega_i = \beta_v^2\langle u_i'(\boldsymbol{x},t)u_3'(\boldsymbol{x},t)\rangle K_1(\beta_{u_iu_3})\frac{\partial\langle T\rangle}{\partial x_3}+\beta_v\beta_\theta\langle u_i'(\boldsymbol{x},t)t'(\boldsymbol{x},t)\rangle K_2(\beta_{u_it}). \tag{B 13}$$

Case 3: When $\partial\langle T\rangle/\partial x_1 \neq 0$ and $\partial\langle T\rangle/\partial x_2 = \partial\langle T\rangle/\partial x_3 = 0$.
In this case, $G_3 = 0$ and

$$G_1(t_1|t) = \int_{t_1}^t \mathrm{e}^{-\beta_\theta(t-s)}\beta_\theta G_{11}(t_1|s)\frac{\partial\langle T\rangle}{\partial x_1}\,\mathrm{d}s, \tag{B 14}$$

$$G_2(t_1|t) = \int_{t_1}^t \mathrm{e}^{-\beta_\theta(t-s)}\left[\beta_\theta G_{21}(t_1|s)\frac{\partial\langle T\rangle}{\partial x_1}+G_{22}(t_1|s)\frac{\partial Q}{\partial x_2}\right]\mathrm{d}s, \tag{B 15}$$

$$\Omega = \beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_1'(\boldsymbol{x},t)\rangle K_1(\beta_{tu_1})\frac{\partial\langle T\rangle}{\partial x_1}+\beta_v\langle t'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle K_1(\beta_{tu_2})\frac{\partial Q}{\partial x_2}$$
$$+A\beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle K_1(\beta_{tu_2})\frac{\partial\langle T\rangle}{\partial x_1}+\beta_v\beta_\theta\langle t'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle L(\beta_{tu_2})$$
$$+\beta_\theta^2\langle t'(\boldsymbol{x},t)t'(\boldsymbol{x},t)\rangle K_2(\beta_{tt}), \tag{B 16}$$

and

$$\omega_i = \beta_v^2\langle u_i'(\boldsymbol{x},t)u_1'(\boldsymbol{x},t)\rangle K_1(\beta_{u_iu_1})\frac{\partial\langle T\rangle}{\partial x_1}+\frac{\beta_v^2}{\beta_\theta}\langle u_i'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle K_1(\beta_{u_iu_2})\frac{\partial Q}{\partial x_2}$$
$$+A\beta_v^2\langle u_i'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle K_1(\beta_{u_iu_2})\frac{\partial\langle T\rangle}{\partial x_1}+\beta_v^2\langle u_i'(\boldsymbol{x},t)u_2'(\boldsymbol{x},t)\rangle L(\beta_{u_iu_2})$$
$$+\beta_v\beta_\theta\langle u_i'(\boldsymbol{x},t)t'(\boldsymbol{x},t)\rangle K_2(\beta_{u_it}). \tag{B 17}$$

Here $A = -2\alpha/\beta_v$ and

$$L(\beta) = \frac{a_v \alpha \beta_\theta}{\beta_v^2} \frac{\partial \langle T \rangle}{\partial x_1} \left\{ \frac{e^{-\beta t}(\beta_\theta t - 1)}{\beta_\theta^2} + \frac{e^{-(\beta_v+\beta)t}(\beta_\theta t - \beta_v t - 1)}{(\beta_\theta - \beta_v)^2} \right.$$
$$\left. + e^{-(\beta_\theta+\beta)t} \left[\frac{1}{\beta_\theta^2} + \frac{1}{(\beta_\theta - \beta_v)^2} \right] \right\}$$
$$+ \frac{\beta_\theta \alpha}{\beta_v} \frac{\partial \langle T \rangle}{\partial x_1} \left\{ \frac{1 - e^{-\beta t}(\beta t + 1)}{\beta_\theta \beta^2} + \frac{1 - e^{-(\beta_v+\beta)t}(\beta_v t + \beta t + 1)}{(\beta_\theta - \beta_v)(\beta_v + \beta)^2} + \frac{e^{-\beta t} - 1}{\beta_\theta^2 \beta} \right.$$
$$\left. + \frac{e^{-(\beta_v+\beta)t} - 1}{(\beta_v + \beta)(\beta_\theta - \beta_v)^2} + \frac{1 - e^{-(\beta_\theta+\beta)t}}{\beta_\theta + \beta} \left[\frac{1}{\beta_\theta^2} + \frac{1}{(\beta_\theta - \beta_v)^2} \right] \right\}. \tag{B 18}$$

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